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There is concern that high power sources of electromagnetic radiation may cause physical harm to an exposed individual even when the frequencies are below those of X rays. Specifically concerns arise in the use of microwave equipment, radars, lasers, active imaging devices and transmitters. Early efforts to address this question considered total absorbed power and then local internal temperature increases; Computer algorithms to make these predictions were developed by USAF/SAM. Recently, as equipment with small duty cycles and very high peak power have been developed, concerns over the effect of sweeping fields and electromagnetic transients on biological structures have arisen. The algorithm developed in this report has as its purpose a highly accurate benchmark code which will predict the response of an N layer bianisotropic spherical structure to multiple plane waves with different amplitudes, frequencies, polarizations, and directions of travel and full wave solutions involving all of the P_n^m.

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Algorithms for the determination of spatial and spectral distribution of electromagnetic energy in a simulated biostructure subjected to transient, spatially heterogeneous radiation with applications to radar hazard assessment and cancer therapy

AFOSR - 90 - 0183

D. K. Cohoon

43 Skyline

Glen Mills, PA 19342

There is concern that military or commercial radars and radiofrequency electromagnetic wave communication equipment, and electron beam lasers may have deleterious biological effects on an exposed individual. The earliest thoughts on the subject of microwave safety standards focused on the concept of total absorbed power as an assessment of a hazard level of a given source making the incorrect assumption that power density distributions within the body of an exposed human were uniform. Calculations ([1], [2], [3]) in the early 1970s at Brooks AFB showed that there were indeed interior hot spots in simulated cranial structures exposed to time harmonic radiation; and these computer calculations were verified experimentally ([3] [4]). One might then think that one could use maximum predicted temperature increase in a realistic model of man ([13]) and the temperature increase that is known to be a danger to cells as a straightforward means of determining the threat to human well being of a source of microwaves; while this would be a good step for this type of source, this is is an extremely difficult calculation and only the methods of ([5]) are known to have a chance of succeeding in modeling the details of the distribution of electromagnetic energy in the heart, liver, spleen, kidneys, et cetera. This method, unlike the moment method, which has never accurately predicted internal field distributions, but only the "more forgiving" bistatic cross sections works like an ordinary differential equation solver in the sense that

one can potentially, by simply doing enough calculation and not using additional memory, improve sufficiently close calculations to machine precision. With the moment method one would be required to "increase the sampling rate" which would mean that more computer memory would be required to approximate the solution; which would eventually make one realize that with the moment method the "realistic model of man" is beyond the capability of existing computers as it is traditionally applied. Even with our more advanced method ([5]) we need highly efficient methods of inverting matrices ([6]), as operations requiring inversion of sufficiently large dense, complex square matrices by traditional operations will have a computational complexity that is beyond the capability of existing computers.

A method that requires no inversion is the resolvent kernel method (Cohoon ??) which has required intensive computational efforts, but if implemented on machines with very many processors may provide an alternative. With the resolvent kernel method described in ([11]) the apriori unknown induced electric fields $\vec{E}$ and and magnetic fields $\vec{H}$ appear outside all integrals, with the only fields to be integrated being the known electric fields $\vec{E}^i$ and and magnetic fields $\vec{H}^i$ of the external stimulating source of electromagnetic radiation. The part that requires real work is the solution of the differential equation involving tensor valued functions needed to obtain the kernels associated with the integral operators appearing in ([11]).

Another concern about solutions of integral equation formulations of electromagnetic interaction problems is the equivalence of the formulation with the two partial differential equations, the Faraday Maxwell equation involving $curl(\vec{E})$ and the Ampere Maxwell equation involving $curl(\vec{H})$ and

the formulation with integral equations ([12]) and the question of uniqueness when formulated in a prescribed function space of allowable electric and magnetic field vectors. The answers to some of these questions in three and seven dimensions is provided in ([12])

However, the subject of this report is another concern, namely the accurate prediction of the response of an N layer spherical structure with tensor electromagnetic properties, with the additional property that both the electric vector $\vec{E}$ and the magnetic vector $\vec{H}$ appear on the right side of the Faraday Maxwell equation and the Ampere Maxwell equation,

- to multiple plane waves with different polarizations and complex amplitudes and frequencies coming from different directions, and to
- electromagnetic waves represented by all the associated Legendre functions P_n^m instead of just the P_n^1 that one sees in the usual Mie solution.

The report with its complete theoretical development and attendant computer calculations ([9]) shows how both of these were carried out successfully. The object here is the concern that even though internal power density distributions induced in a biological structure by a radar or other emitting installation is quite low that the sweeping nature of the beam might interfere with natural biochemical cycles and thereby alter biochemical reactions. There is also a concern that a complex mix of frequencies and electromagnetic fields might be absorbed by an individual in a highly efficient manner, thereby producing an unexpectedly high internal field distribution.

1 Computational Capability

The primary 12K line computer code ([9]) delivered to the government has the ability to

- calculate the near or far field E or H plane bistatic cross section,
- plot the internal power density distribution along the direction of propagation of a time harmonic plane wave impinging on the N layer structure,
- calculate the absorbed and reflected power as a function of any electrical property in any layer or the frequency,
- calculate the absorption efficiency and the scattering efficiency and the extinction efficiency as a function of wavelength and geometry,
- calculate the power scattered into any solid angle as a function of any electrical property in any layer,
- calculate the power density distribution in a slice of the N layer sphere passing through the origin with any prescribed normal,
- create a 2 D plot of the Mueller matrix as a function of scattering angle,
- create a 3 D plot of any Mueller matrix entry as a function of scattering angle and interrogating wavelength,

- produce 3 D plots of the square of the length of the magnetic vector, the electric vector, or the Poynting vector for complex impinging waves at any time on any slice of the N layer sphere,

- plot the phase of the components of electric vector of the scattered electromagnetic radiation as a function of position outside the N layer structure

- automatically check the validity of the calculations by producing tables of values of the tangential components of the electric and magnetic vectors on both sides of the separating spheres,

- automatically check the validity of the representation of the incoming radiation by Tesseral harmonics by comparing the exact formula representation with the spherical harmonic expansion, and

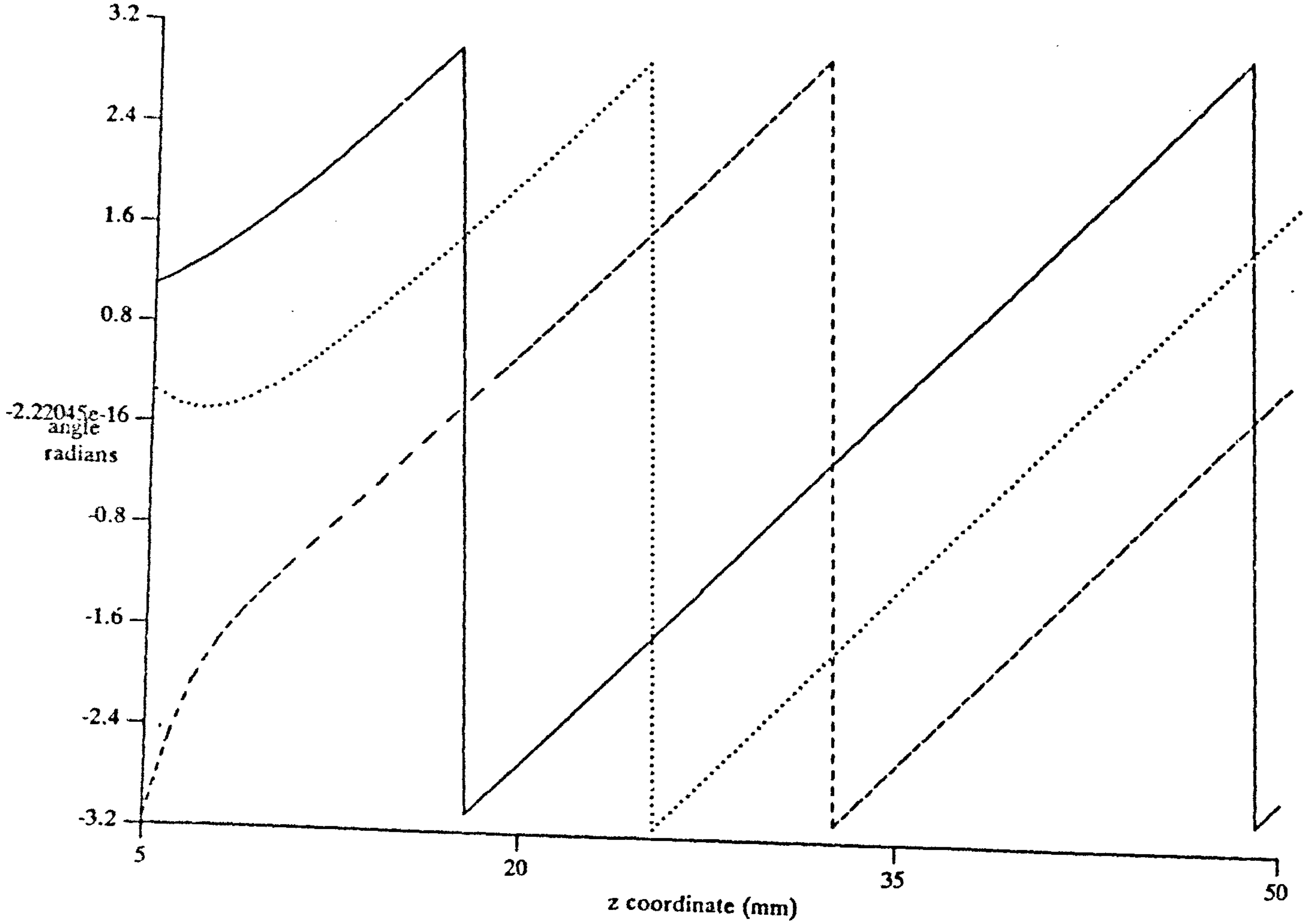
- automatically check the validity of calculations of the inverses of the 4 by 4 matrices used in the bianisotropy analysis by multiplying the computer program calculated inverse by the original matrix and comparing this with the 4 by 4 identity, and

- automatically check the validity of the overall calculation by using volume integration to compute the total absorbed power through the integration of the power density distribution and then comparing this result with the integration of the Poynting vector over the outermost sphere.

A more general structure that permits one to benchmark general pur-

Phases of E vector components

Frequency = 1000 Megahertz, Radius = 5.0 centimeters
Total absorbed power = 1.08681457E-5 Watts
ϵ = 59.86
μ radial = 1+ 0.0i, and μ tangential = 1+0.0i
σ = 1.00152 mhos per meter
Coupling constant is .00 reciprocal meters



pose codes to realistically model man in a microwave field are oblate and prolate spheroids ([7]) and ([8]) and bodies of revolution which include finite cylinders and toroids. Precise calculations of the latter may help one develop an efficient fusion reactor by optimizing the absorption efficiency of the plasma in which the fusion reaction is taking place.

There is a possibility that high energy sources or regular sources ([10]) could physically disturb or shock biological structures or set up waves within bone structures which might over a period of time have a biological effect. The paper ([10]) gives a full semigroup theory of nonlinear electromagnetic responses which could be used in laser surgery and as well describes the energy equation for a stimulated Voigt solid.

1.1 Applications with general societal benefit

The computer calculations ([9]) show that hot spots can be moved about the body by controling the external fields. With precise calculations one could treat cancer by raising the temperature of the tumor by 4 degrees centegrade while not raising the nearby normal tissue to that extent. This would kill a localized tumor without harming the nearby normal tissue; considerable effort and dedication would be required to build a successful device, but the theory is laid down in ([5]).

2 An Exact Microwave Heating Response Formula

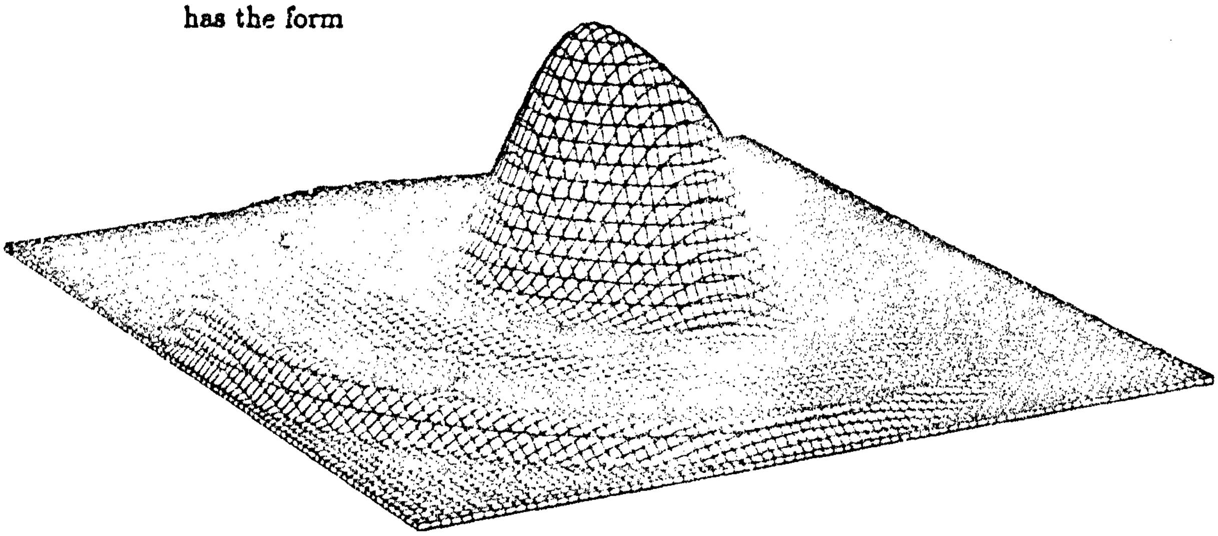
An exact formula for the thermal response of an N layer spherically symmetric structure to electromagnetic radiation has been determined. The thermal conductivity, K, the permittivity, ϵ, the magnetic permeability, μ, and the electromagnetic conductivity, σ, are assumed to be tensors.

2.1 Nonhomogeneous Heat Equation

In this paper we extend previous results ([3]) and give an exact solution to the problem of describing by exact formula the thermal response to low energy electromagnetic radiation of an N layer electromagnetically bianisotropic structure, or said differently a spherically symmetric structure where the permittivity ϵ and magnetic permeability, μ, and the complex electromagnetic conductivity σ are tensors, and which is also thermally anisotropic in the sense that the thermal conductivity, K, is also a tensor. The heat transfer equation is written in the form,

$$\rho c \frac{\partial u}{\partial t} = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\right)\left(r^2 K_r \frac{\partial u}{\partial r}\right) + \frac{1}{r sin(\theta)}\left(\frac{\partial}{\partial \theta}\right)\left(K_\theta \frac{sin(\theta)}{r}\frac{\partial u}{\partial \theta}\right) + \frac{1}{r sin(\theta)}\left(\frac{\partial}{\partial \phi}\right)\left(K_\phi \frac{1}{r sin(\theta)}\frac{\partial u}{\partial \phi}\right) - bu + S \tag{2.1.1}$$

where if the magnetic permeabilities μ and the permittivities ϵ, and the electromagentic conductivity σ are diagonal tensors in the spherical coordinate system and their ϵ and θ components are equal, then the source term has the form



Distribution of power density in a section of a brain tissue sphere exposed to 1 Gigahertz radiation. Relative permittivity is 59.36 and conductivity is 1.00152 mhos per meter.

This shows the potential hazard of internal hot spots and the possibility of treating cancer by raising the temperature of the tumor by 4 degrees C.

$$S = F_0[\omega Im(\mu)(| H_\theta |^2 + | H_\phi |^2) + \omega Im(\mu_r)(| H_r |^2) +$$
$$\omega Im(\epsilon)(| E_\theta |^2 + | E_\phi |^2) + \omega Im(\epsilon_r)(| F \ |^2) +$$
$$Re(\sigma)(| E_\theta |^2) + (| E_\phi |^2)] \tag{2.1.2}$$

where in the case of bianisotropy there are additional terms involving the product of electric and magnetic vector components times coupling terms and where F_0 is a factor for converting MKS energy densities to centimeter gram second(cgs) units of calories per gram degree Centigrade that is given by,

$$F_0 = \frac{1}{2 \times 10^6 \times 4.184} \tag{2.1.3}$$

In order to make use of Legendre functions with integer index in our solution we assume that

$$K = K_\theta \tag{2.1.4}$$

and

$$K = K_\phi \tag{2.1.5}$$

We will make use of the Legendre functions $P_n^m(cos(\theta))$ and note that they satisfy the ordinary differential equation

$$\frac{1}{sin(\theta)} \frac{d}{d\theta} P_n^m(cos(\theta)) =$$
$$\frac{m^2}{sin^2(\theta)} P_n^m(cos(\theta)) - n(n+1) P_n^m(cos(\theta)) \tag{2.1.6}$$

which is equivalent to

$$\frac{d}{dz}((1-z^2)\frac{dW}{dz}) + (n(n+1) - \frac{m^2}{1-z^2})W \; = \; 0 \tag{2.1.7}$$

where

$$z \; = \; cos(\theta) \tag{2.1.8}$$

since (2.1.7) implies that

$$\frac{dv}{d\theta} \; = \; \frac{dv}{dz}\frac{dz}{d\theta} \tag{2.1.9}$$

which implies that

$$\frac{dv}{d\theta} \; = \; -sin(\theta)\frac{dv}{dz} \tag{2.1.10}$$

(Hochstadt [14], page 164). To see this more clearly note that

$$\frac{d}{dz}\left((1-z^2)\frac{dW}{dz}\right) =$$

$$-2cos(\theta)\left(\frac{-1}{sin(\theta)}\frac{dW}{d\theta}\right) + sin^2(\theta)\left\{\frac{-cos(\theta)}{sin^3(\theta)}\right\} + \frac{d^2W}{d\theta^2} \tag{2.1.11}$$

Therefore, we see that

$$\frac{d}{dz}\left((1-z^2)\frac{dW}{dz}\right) = \frac{cos(\theta)}{sin(\theta)}\frac{dW}{d\theta} + \frac{d^2W}{d\theta^2} \tag{2.1.12}$$

This implies that

$$\frac{d}{dz}\left((1-z^2)\frac{dW}{dz}\right) \; = \; \frac{1}{sin\theta}\frac{d}{d\theta}\left(sin(\theta)\frac{dW}{d\theta}\right) \tag{2.1.13}$$

Having developed this understanding of the formulation of Legendre's differential equation we proceed to define the finite Legendre transform (not the integral over the index) by the rule,

$$\mathcal{L}_n^m v = (2n+1)\frac{(n-m)!}{(n+m)!}\int_0^\pi v(\theta)P_n^m(cos(\theta))sin(\theta)d\theta \tag{2.1.14}$$

for m being positive and further restricted by the relation,

$$m \epsilon \{0, 1, ..., n\} \cup \{-1, -2, ..., -n\} \tag{2.1.15}$$

and

$$\mathcal{L}_n^0 v = \left(\frac{2n+1}{2}\right) \int_0^\pi v(\theta) P_n(cos(\theta)) sin(\theta) d\theta \tag{2.1.16}$$

In carrying out the simplification of the heat transfer equation we need in addition to the finite Legendre transform, a finite cosine transform defined by the rule,

$$\mathcal{C}_m u = \frac{1}{\pi} \int_{-\pi}^{\pi} cos(m\phi) u(\phi) d\phi \tag{2.1.17}$$

A calculation shows that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} cos(m\phi) \frac{\partial^2 u}{\partial \phi^2}(\phi) d\phi = \frac{m}{\pi} \int_{-\pi}^{\pi} sin(m\phi) \frac{\partial u}{\partial \phi} d\phi$$

$$= -m^2 \mathcal{C}_m u \tag{2.1.18}$$

We now develop a formula that enables us to simplify the energy equation by successive application of the finite cosine transform $\mathcal{C}_m$ and the finite Legendre transform $\mathcal{L}_m^n$. Using Legendre's differential equation and integration by parts we find that

$$\mathcal{L}_n^m \mathcal{C}_m \left(\frac{1}{r sin(\theta)} \frac{\partial}{\partial \theta} \left(K sin(\theta) \frac{1}{r} \frac{\partial u}{\partial \theta}\right)\right) +$$

$$\mathcal{L}_n^m \mathcal{C}_m \left(\frac{1}{r sin(\theta)} \frac{\partial}{\partial \phi} \left(K \frac{1}{r sin(\theta)} \frac{\partial u}{\partial \phi}\right)\right) = \frac{-n(n+1)}{r^2} K \mathcal{L}_n^m \mathcal{C}_m u \tag{2.1.19}$$

since the terms involving m^2 cancel out if the two tangential components of the thermal conductivity tensor are equal.

We now apply the combined finite Legendre and finite cosine transforms to all terms of the energy equation,

$$\rho c \frac{\partial}{\partial t}(\mathcal{L}_n^m \mathcal{C}_m u) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 K_r \frac{\partial}{\partial r}(\mathcal{L}_n^m \mathcal{C}_m u)\right) +$$

$$\frac{-n(n+1)}{r^2} K \mathcal{L}_n^m \mathcal{C}_m u - \mathcal{L}_n^m \mathcal{C}_m bu + \mathcal{L}_n^m \mathcal{C}_m S \qquad (2.1.20)$$

We simplify the writing of the above equation by introducing the variables

$$U_{(m,n)} = \mathcal{L}_n^m \mathcal{C}_m u \qquad (2.1.21)$$

and

$$\mathcal{S}_{(m,n)} = \mathcal{L}_n^m \mathcal{C}_m S \qquad (2.1.22)$$

we, therefore, see that the original energy equation may be transformed into the relation,

$$\rho c \frac{\partial}{\partial t}(U_{(m,n)}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 K_r \frac{\partial}{\partial r}\left(U_{(m,n)}\right)\right) +$$

$$\frac{-n(n+1)}{r^2} K U_{(m,n)} - bU_{(m,n)} + \mathcal{S}_{(m,n)} \qquad (2.1.23)$$

We create another finite transform with respect to the radial variable by making use of the oscillation theorem to select a series of radial eigenfunctions, $Z_{(n,k)}(r)$, satisfying

$$(\lambda_{(n,k)} r^2 \rho c - Kn(n+1) - br^2) Z_{(n,k)} + \frac{d}{dr}\left(r^2 K_r(r) \frac{d}{dr} Z_{(n,k)}(r)\right) = 0. \quad (2.1.24)$$

the regularity conditions that state that

$$K_r \frac{d}{dr} Z_{(n,k)}(r) \epsilon \mathcal{C} \qquad (2.1.25)$$

and

$$Z_{(n,k)} \epsilon C \tag{2.1.26}$$

where C denotes the space of continuous functions on the real line with the origin removed. We now multiply all terms of our transformed equation (2.1.23) by the $Z_{(n,k)}$ which satisfies (2.1.24) and integrate from 0 to R_N, which is the radius of the outermost sphere in our N layer structure. Upon doing so we obtain the relation,

$$\int_0^{R_N} Z_{(n,k)}(r)\frac{\partial}{\partial t}\left(U_{(m,n)}\right)\rho c r^2 dr =$$

$$\int_0^{R_N} Z_{(n,k)}(r)\frac{\partial}{\partial r}\left(r^2 K_r \frac{\partial}{\partial r}U_{(m,n)}\right) dr - \int_0^{R_N} Z_{(n,k)}(r) K n(n+1) U_{(m,n)}(r) dr$$

$$- \int_0^{R_N} b U_{(m,n)} r^2 Z_{(n,k)}(r) dr + \int_0^{R_N} r^2 S_{(m,n)}(r,t) Z_{(n,k)}(r) dr \tag{2.1.27}$$

We use integration by parts to simplify equation (2.1.27), and we find that

$$\int_0^{R_N} Z_{(n,k)}(r)\frac{\partial}{\partial r}\left(r^2 K_r \frac{\partial}{\partial r}U_{(m,n)}(r,t)\right) dr =$$

$$Z_{(n,k)}(R_N) R_N^2 K_r(R_N)\frac{\partial}{\partial r}U_{(m,n)}(R_N,t) -$$

$$\int_0^{R_N} r^2 K_r \frac{\partial}{\partial r}U_{(m,n)}\left(\frac{d}{dr}Z_{(n,k)}(r)\right) dr$$

$$= Z_{(n,k)}(R_N) R_N^2 K_r(R_N)\frac{\partial}{\partial r}U_{(m,n)}(R_N,t)$$

$$- R_N^2 K_r(R_N) Z^{(1)}_{(n,k)}(R_N) U_{(m,n)}(R_N,t)$$

$$+ \int_0^{R_N} U_{(m,n)}(r,t)\frac{d}{dr}\left(r^2 K_r \frac{d}{dr}Z_{(n,k)}\right) dr \tag{2.1.28}$$

Observe that the functions $V = U_{(m,n)}$ or $V = Z_{(n,k)}$ satisfy exactly the same boundary condition, namely the Newton cooling law constraint given by

$$K_r V^{(1)}(R_N) + HV(R_N) = 0 \tag{2.1.29}$$

and upon replacing the derivative terms by using this Newton cooling law relation we see that

$$\begin{aligned} & Z_{(n,k)}(R_N)R_N^2 K_r(R_N)U^{(1)}_{(m,n)}(R_N,t) - \\ & R_N^2 K_r(R_N)Z^{(1)}_{(n,k)}(R_N)U_{(m,n)}(R_N,t) \\ & = Z_{(n,k)}(R_N)R_N^2(-HU_{(m,n)}) \\ & - R_N^2(-HZ_{(n,k)})(R_N)U_{(m,n)} = 0 \end{aligned} \tag{2.1.30}$$

Thus, we find that

$$\begin{aligned} & \int_0^{R_N} Z_{(n,k)}(r)\frac{\partial}{\partial r}\left(r^2 K_r \frac{\partial}{\partial r}U_{(m,n)}(r,t)\right) dr \\ & = \int_0^{R_N} U_{(m,n)}(r,t)\frac{d}{dr}\left(r^2 K_r \frac{d}{dr}Z_{(n,k)}(r)\right) dr \end{aligned} \tag{2.1.31}$$

Substituting equation (2.1.23) into this relationship yields

$$\begin{aligned} & \int_0^{R_N} Z_{(n,k)}(r)\frac{\partial}{\partial r}\left(r^2 K_r \frac{\partial}{\partial r}U_{(m,n)}(r,t)\right) dr = \\ & \int_0^{R_N} U_{(m,n)}\left\{Kn(n+1) + br^2 - \lambda_{(n,k)}r^2\rho c\right\} Z_{(n,k)}(r)dr \end{aligned} \tag{2.1.32}$$

Using these relationships in equation (2.1.28) we find that

$$\int_0^{R_N} \frac{\partial}{\partial t}U_{(m,n)}(r,t)Z_{(n,k)}(r)\rho c r^2 dr =$$

$$-\lambda_{(n,k)}\int_0^{R_N} U_{(m,n)}(r,t)Z_{(n,k)}(r)\rho c r^2 dr$$

$$+\int_0^{R_N} \mathcal{S}_{(m,n)}(r,t)Z_{(n,k)}(r)r^2 dr \tag{2.1.33}$$

Thus, if we define the finite radial transform, $\mathcal{T}$

$$\mathcal{T}_{(n,k)}(f) = \frac{\int_0^{R_N} f(r)Z_{(n,k)}(r)\rho c r^2 dr}{\int_0^{R_N} Z_{(n,k)}(r)^2\rho c r^2 dr} \tag{2.1.34}$$

then if we let

$$b_k^{(m,n)}(t) = \mathcal{T}_{(n,k)}\mathcal{L}_n^m\mathcal{C}_m\left(\frac{\mathcal{S}}{\rho c}\right)(t) \tag{2.1.35}$$

we see that

$$\frac{d}{dt}\mathcal{T}_{(n,k)}U_{(m,n)} + \lambda_{(n,k)}\mathcal{T}_{(n,k)}U_{(m,n)} = b_k^{(m,n)}(t) \tag{2.1.36}$$

so that if we define the expansion coefficient by the rule,

$$a_k^{(m,n)}(t) = \mathcal{T}_{(n,k)}\mathcal{L}_n^m\mathcal{C}_m u \tag{2.1.37}$$

then the problem reduces to that of solving the ordinary differential equation

$$\frac{d}{dt}a_k^{(m,n)}(t) + \lambda_{(n,k)}a_k^{(m,n)}(t) = b_k^{(m,n)}(t) \tag{2.1.38}$$

The solution of this differential equation is given by

$$a_k^{(m,n)}(t) = \int_0^t exp(\lambda_{(n,k)}(t-\tau))b_k^{(m,n)}(\tau)d\tau \tag{2.1.39}$$

where the term $b_k^{(m,n)}$ is defined by equations (2.1.2) and (2.1.35).

For a variety of pulsed heating schemes ([4]), this integral has been evaluated exactly. In doing so we have made use of the dramatic difference in the time scales of conductive and radiative transfer in the materials being irradiated.

2.2 THE RADIAL FUNCTIONS

We have seen that the radial functions satisfy the ordinary differential equation,

$$\frac{d}{dr}\left(r^2\frac{d}{dr}Z_{(n,k)}\right)+$$
$$\left\{\frac{\lambda_{(n,k)}\rho c-b}{K_r}-\frac{n(n+1)K}{K_r}\right\}Z_{(n,k)}=0 \qquad (2.2.1)$$

We let

$$z^2=\left(\frac{\lambda_{(n,k)}\rho c-b}{K_r}\right)r^2 \qquad (2.2.2)$$

in each layer. Then equation (2.2.1) and equation (2.2.2) imply that if

$$W(z)=Z_{(n,k)}(r) \qquad (2.2.3)$$

that then

$$z^2W^{(2)}(z)+\frac{2}{z}W^{(1)}(z)+\left(1-\frac{n(n+1)K}{z^2K_r}\right)W=0 \qquad (2.2.4)$$

Rearranging the terms involving the derivative we find that

$$\frac{1}{z}\left(\frac{d}{dz}\right)^2(zW(z))+\left(1-\frac{n(n+1)K}{z^2K_r}\right)W \qquad (2.2.5)$$

Thus, the radial functions are given by

$$W=\Psi_\nu(z)=\frac{\sqrt{\pi}J_{\nu+1/2}(z)}{\sqrt{2}\sqrt{z}} \qquad (2.2.6)$$

where

$$\nu=-1/2+\sqrt{\frac{1}{4}+\frac{n(n+1)K}{K_r}} \qquad (2.2.7)$$

2.3 EIGENVALUE DETERMINATION

We are seeking by computer algorithm a solution of an ordinary differential equation,

$$\frac{d}{dr}\left(r^2\frac{d}{dr}Z_{(n,k)}\right) +$$

$$\left\{\frac{\lambda_{(n,k)}\rho c - b}{K_r} - \frac{n(n+1)K}{K_r}\right\} Z_{(n,k)} = 0 \qquad (2.3.1)$$

which has a singular point at $r = 0$, and piecewise smooth coefficients with the additional property that one of the coefficients depends on a parameter which we have denoted by $\lambda_{(n,k)}$, where the solution, $Z_{(n,k)}(r)$ and the product of the radial conductivity K_r and the derivative of $Z_{(n,k)}$ are continuous and

$$\lim_{r\to\infty} \left(K_r(r)\frac{d}{dr}Z_{(n,k)}(r) + HZ_{(n,k)}(r)\right) = 0 \qquad (2.3.2)$$

These eigenvalues and eigenfunctions are obtained by a shooting method by defining $Z(r, \lambda)$ to be the solution of the ordinary differential equation.

$$\frac{d}{dr}\left(r^2\frac{d}{dr}Z(r,\lambda)\right) +$$

$$\left\{\frac{\lambda\rho c - b}{K_r} - \frac{n(n+1)K}{K_r}\right\} Z(r,\lambda) = 0 \qquad (2.3.3)$$

which has an integrable singularity at $r = 0$ and which satisfies the regularity conditions and the Newton cooling law boundary conditions at the outer boundary of the sphere.

2.4 RADIATIVE HEATING

Because of orthonormality of the transformations we find that the difference u between the induced thermal excursion and the ambient temperature in the irradiated solid is given by

$$u(r,\theta,\phi,t)=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{n}a_k^{(m,n)}(t)P_n^m(cos(\theta))cos(m\phi)Z_{(n,k)}(r) \qquad (2.4.1)$$

where the expansion coefficients are determined by equation (2.1.39), and the heat source term is given by equation (2.1.2). In the computer code the expansion coefficients are precomputed and saved. Once the expansion coefficients are known, the microwave induced thermal excursion can be computed at thousands of points for a modest computer cost. The entire code runs on a personal computer.

In the following sample computation, which gives a comparison of radiation induced temperature increases measured with a Vitek non field perturbing thermal probe and predictions of the computer program, a dielectric ball was enclosed in styrofoam. The calculation is for an isotropic muscle equivalent spherical structure with a radius of 3.3 centimeters exposed to 1.2 Gigahertz continuous wave radiation with a power of 70 milliwatts per square centimeter. The time of exposure for figure 1 was 30 seconds. In Figure 2 which follows the Vitek probe was placed at the center of the structure and the the power was turned off after 5.5 minutes.

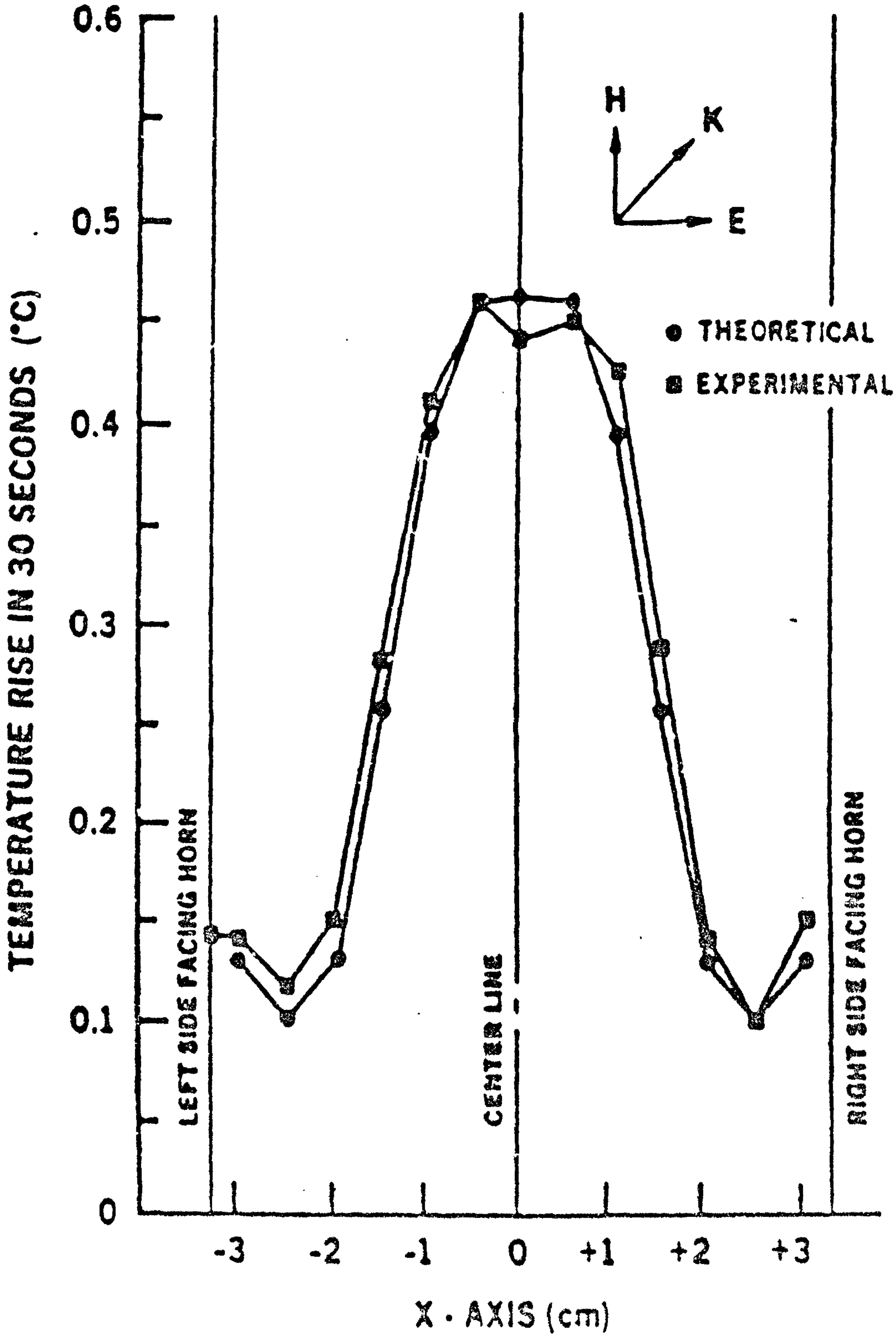


Figure 1. A 3.3 centimeter radius spherical structure exposed to 1.2 Gigahertz with a power of 70 milliwatts per square centimeter for 30 seconds. The structure is isotropic and the relative permittivity is 50.4 and the electromagnetic conductivity is 1.52 mhos per meter. The specific heat is .84 calories per gram degree Centigrade. The thermal conductivity in calories per centimeter per second per degree Centigrade is .0012

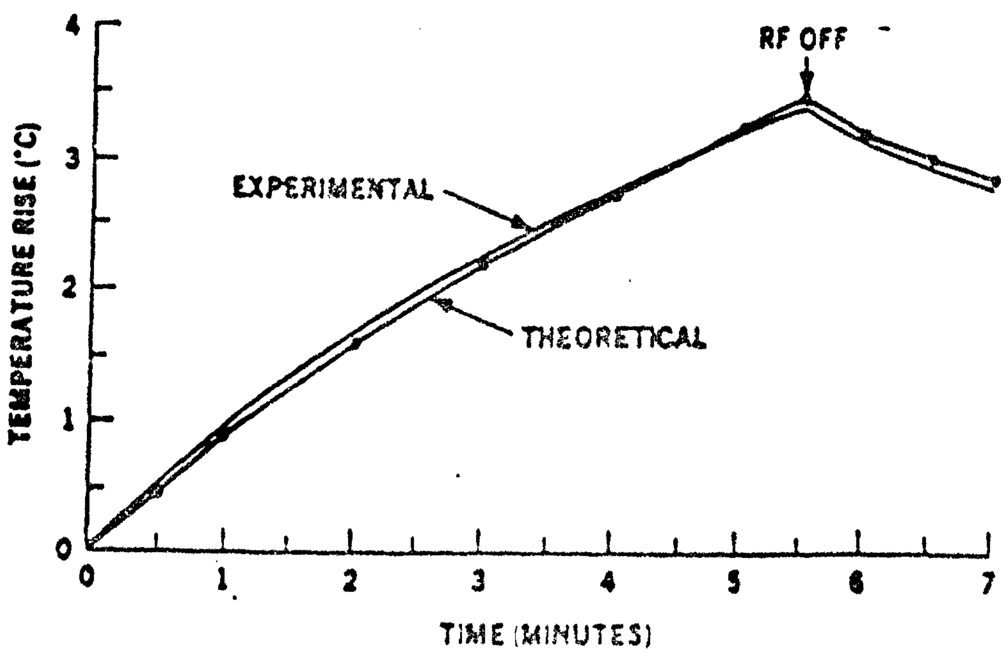


Figure 2. The temperature at the center of a 3.3 centimeter radius spherical structure exposed to 1.2 Gigahertz radiation with a power of 70 milliwatts per square centimeter as a function of time. The power was turned off after 5.5 minutes. The thermal and electromagnetic parameters are the same as those of Figure 1. The Newton cooling law constant used at the surface is

$$H = 5.722 \times 10^{-5} \text{ calories/cm}^2\text{/degree}^{\circ}\text{C/second}$$

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Exact Mie like determination of the response of an N layer bianisotropic structure with regions of continuity of tensorial electromagnetic properties delimited by concentric spheres to multiple plane waves and general full wave radiation with complex spatial and temporal patterns

D. K. Cohoon

March 7, 1992

We are interested in assessing the level of biological hazard posed by complex sources of electromagnetic radiation. A part of this effort is to understand in space and time how fields and energy densities and flows change in the interior of a multitissue simulated biological structure, and to have a highly accurate benchmark for detailed models of man in an electromagnetic field.

We consider the problem of determining the optical and absorption efficiency of a class of N layer full tensor electromagnetically bianisotropic spheres to a plane polarized electromagnetic radiation. Considerable flexibility ([47]) has been demonstrated with two

layer structures in relating their properties in such a way that these particles have an extremely high optical and absorption efficiency. Groundwork has been laid for the design of materials with differing electromagnetic properties in different directions which have extremely high efficiencies of absorption.

By careful analysis it would be possible to do the same for a heterogeneous radiation source(Barton [8], Chevaillier [19] [18], Chylek [20] Schaub [42], Tsai [46], Yeh [57]) when these spheres are placed in an ambient medium with material properties such that if Ω is an open set in the ambient medium and

$$\int_\Omega div(\vec{E} \times \vec{H}^*)dv = 0,$$

then $\vec{E}$ and $\vec{H}$ are both zero in Ω

In this paper we describe the exact solution to the problem of describing the interaction of electromagnetic radiation with an N layer structure whose regions of continuity of tensorial electromagnetic properties are separated by concentric spheres. We assume that each of the layers are bianisotropic. For the most general case, the radial functions are solutions of a system of equations, and we get a four parameter family for each index in each layer. However, we also get an interesting, but easily computerizable example.

Bianisotropic materials have used (Ferencz [25], Gamo [26], Hebenstreit [29], Shiozawa, [44] and Yeh [56]) in modeling a medium moving through an electromagnetic field. We consider also the possibility of an electromagnetic field whose spatial distribution would suggest a complex source that would include an off center laser beam interaction with a droplet or a radar beam sweeping across a stationary structure. By considering a layered spherically symmetric structure whose core may be metallic and with outer layers having complex material properties or containing sources of radiation, we may be able to predict the level of the hazard experienced by an individual with a metalic bone replacement or clamp who is placed in such a field.

The source of internal power density distribution for a bianisotropic structure exposed to external sources is distinct from anisotropic materials, since terms involving the product of the electric vector $\vec{E}$ and the magnetic vector $\vec{H}$ appear in the internal power density distribution. Using the concepts contained in this paper, a solution of an energy equation

with a tensor conductivity can be obtained by an exact formula when the electromagnetic properties do not change during the exposure process. Using the derived energy density distribution as a source term, a more general nonlinear heat equation, taking into account radiative conductivity concepts can be derived. Several authors (Barton [8], Chylek [22], Schaub [42]) simply assume that the power density depends on the square of the length of the electric vector times the conductivity. In a bianisotropic material, however, there is a power density contribution from the coupling of the electric and magnetic vectors (see equation (5.2.6).

Contents

1 A Mie Like Solution for Bianisotropic Sphere Scattering

1.1 Introduction

Although it is possible to develop an integral equation formulation of the problem of describing the scattering of electromagnetic radiation by a bounded three dimensional body (Jones [33], pp 528-529), the only bounded body for which a truly exact solution has been obtained to the problem for describing its response to electromagnetic radiation have been those with spherical symmetry. It is possible to give a representation (Jones [33], pp 490 to 495) of the fundamental Green's tensor $\overline{\overline{\Gamma}}$ satisfying

$$curl(curl(\overline{\overline{\Gamma}})) - k^2\overline{\overline{\Gamma}} = \overline{\overline{I}}\delta \tag{1.1.1}$$

in terms of vector spherical harmonics and to use these to develop a concise derivation of the solution of the problem of describing scattering by a sphere (Jones [33] pp 496-526). Some earlier work on anisotropic sphere scattering ([28] [32]), [49], [53]) have extended the classical result of Mie ([34] 1908) which is believed to have been first obtained by Clebsch

([23] 1863). We describe, here, an exact Mie like solution that is applicable to a class of bianisotropic spheres.

1.2 Problem Definition

We assume that $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$ are tensors defining the permeability and permittivity that are functions of the spatial variables and the frequency ω of the radiation. Here Maxwell's equations have the form

$$curl(\vec{E}) = -i\omega\bar{\bar{\mu}}\vec{H} - \bar{\bar{\alpha}}\vec{E} \tag{1.2.1}$$

and

$$curl(\vec{H}) = i\omega\bar{\bar{\epsilon}}\vec{E} + \bar{\bar{\sigma}}\vec{E} + \bar{\bar{\beta}}\vec{H} \tag{1.2.2}$$

In the ambient medium we assume that the tensors $\bar{\bar{\alpha}}$ and $\bar{\bar{\beta}}$ are the zero tensor $\bar{\bar{0}}$. In this paper the energy balance is described which enables us to validate a computer code for describing the interaction of radiation with an N layer bianisotropic sphere where the layers may be separated by impedance sheets. The inner core may be penetrable or perfectly conducting.

1.3 Spherical Harmonics and Orthogonality Relations

The basic idea of the code is that the induced and scattered electric and magnetic vectors can be expressed in terms of

$$\vec{A}_{(m,n)} = \left[im\frac{P_n^m(cos(\theta))}{sin(\theta)}\vec{e}_\theta - \frac{d}{d\theta}P_n^m(cos(\theta))\vec{e}_\phi\right]exp(im\phi), \tag{1.3.1}$$

$$\vec{B}_{(m,n)} = \left[\frac{d}{d\theta}P_n^m(cos(\theta))\vec{e}_\theta + im\frac{P_n^m(cos(\theta))}{sin(\theta)}\vec{e}_\phi\right]exp(im\phi), \tag{1.3.2}$$

and

$$\vec{C}_{(m,n)} = P_n^m(cos(\theta))exp(im\phi)\vec{e}_r, \tag{1.3.3}$$

where $\vec{e}_r$, $\vec{e}_\theta$, and $\vec{e}_\phi$ are the unit vectors perpendicular, respectively, to the $r = 0$, $\theta = 0$, and $\phi = 0$, coordinate planes, and where $P_n(cos(\theta))$ is the ordinary Legendre function

defined by Rodrigues's formula

$$P_n(z) = \frac{1}{2^n n!}\left(\frac{d}{dz}\right)^n (z^n - 1)^n \tag{1.3.4}$$

The associated Legendre functions P_n^m are given by

$$P_n^m(z) = \left(1 - z^2\right)^{m/2}\left(\frac{d}{dz}\right)^m P_n(z) \tag{1.3.5}$$

It is obvious that even without integrating over a sphere that the dot product of either of $\vec{A}_{(m,n)}$ or $\vec{B}_{(m,n)}$ with $\vec{C}_{(m,n)}$ is zero. The orthogonality of the functions $exp(im\phi)$ and $\exp(i\tilde{m}\phi)$ on the unit circle for $m \neq \tilde{m}$ show that if as in ([11]) we define the inner product of two vector valued functions $\vec{U}(\theta,\phi)$ and $\vec{U}(\theta,\phi)$ defined on the unit sphere by,

$$<\vec{U},\vec{V}> \;=\; \int_0^{2\pi}\int_0^{\pi} \vec{U}(\theta,\phi)\cdot\vec{V}(\theta,\phi)^* sin(\theta)d\theta d\phi \tag{1.3.6}$$

with two different values of m are orthogonal. If we take the dot product of two distinct members of the collection

$$S = \left\{\vec{A}_{(m,n)}, \vec{B}_{(m,n)}, \vec{C}_{(m,n)} : m\in Z, \text{ and } n\in\{|\,m\,|, |\,m\,|+1,\cdots\}\right\}, \tag{1.3.7}$$

with the same values of m and make use of ([2], p 333) the negative index relationship

$$P_\nu^{-\mu}(z) \;=\; \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left[P_\nu^\mu(z) - \frac{2}{\pi}exp(-i\mu\pi)sin(\mu\pi)Q_\nu^\mu(z)\right] \tag{1.3.8}$$

we find that any two members with different values of n are orthogonal with respect to the inner product defined by equation (1.3.6). For example, to see that

$$<\vec{A}_{(m,n)}, \vec{B}_{(m,r)}> \;=\; 0 \tag{1.3.9}$$

for all n and r we note that this dot product reduces to

$$im(2\pi)\int_0^{\pi}\frac{d}{d\theta}\left[P_n^m(cos(\theta))P_r^m(cos(\theta))\right]d\theta \;=\; im(2\pi)\int_{-1}^{1}\frac{d}{dx}\left\{P_n^m(x)P_r^m(x)\right\}dx \tag{1.3.10}$$

The details of the remaining orthogonality relations are found in ([11]) or can be derived from properties of the Legendre functions described in Jones ([33]).

Plane waves in free space can be represented using the functions described above by carrying out the expansion(Bell, [10] page 51 and Jones [33], page 490, equation 94)

$$exp(-ik_0 r cos(\theta)) = \sum_{n=0}^{\infty} a_n P_n(cos(\theta)) j_n(k_0 r) \tag{1.3.11}$$

where the expansion coefficients a_n are given by (see Jones [33], page 490)

$$a_n = (-i)^n(2n+1). \tag{1.3.12}$$

These coefficients are determined by letting $z = k_0 r$, carrying out a Taylor series expansion in z, and making use of the orthogonality relationships

$$\int_0^{\pi} P_n(cos(\theta))P_m(cos(\theta)sin(\theta)d\theta = \begin{cases} 2/(2n+1) & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{1.3.13}$$

This equation is based on the relation (Bell [10], page 61)

$$\int_{-1}^{1}(z^2-1)^n dz = \int_{-1}^{1}(z-1)^n(z+1)^n dz = \frac{2^{2n+1}(n!)^2}{(2n+1)!}\cdot(-1)^n \tag{1.3.14}$$

which follows from integration by parts in the left side of equation (1.3.13). This relationship can be proven using the Rodrigues definition (equation 1.3.4). By using the notion that the algebraic structure formed by linearly combining these vector fields in a ring of radial functions is invariant under the curl operation also enables one to get an exact solution to the scattering problem for bianisotropic spheres.

1.4 Plane Wave Spectral Decomposition

An alternative to the consideration of a full wave solution is the utilization of a plane wave spectral decomposition, where specially selected plane waves with carefully chosen (i) amplitudes, (ii) polarizations, and (iii) frequencies are used to represent a complex impinging wave. Two calculations are shown here. They represent the response of a sphere of brain tissue to a complex radiation field. For the purpose of illustration we give plots of the real part of the radial component of the electric vector on the intersection of this sphere of brain tissue with a plane whose normal coincides with the laboratory z-axis. The two following plots show the real part of the radial component of the electric vector on this slice at two different times.

Figure 1.4.1. This figure shows the real part of the radial component of the electric vector in a slice of a sphere subjected two plane waves coming from two different directions. The time of observation is .3 times the common period of the two waves after a reference time.

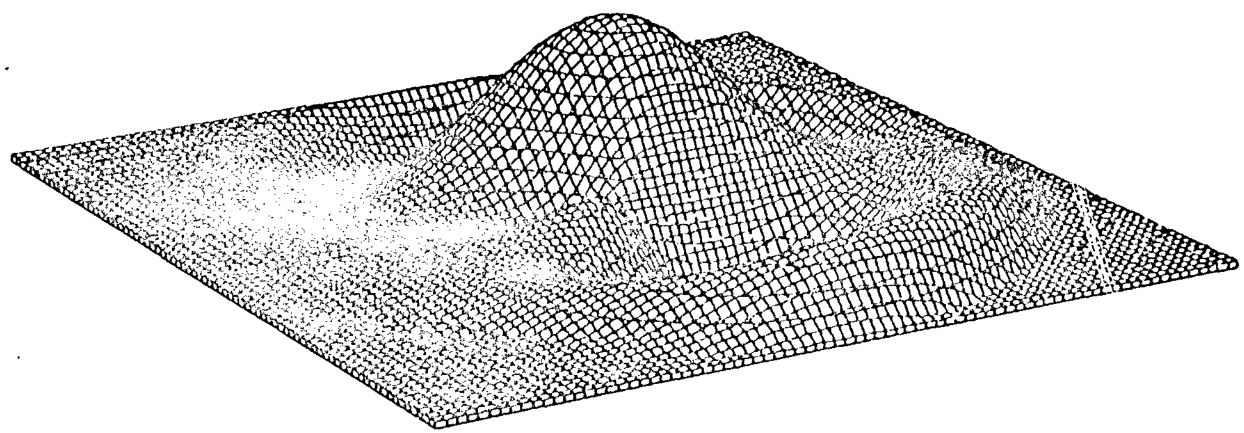


Figure 1.4.2. This figure shows the real part of the radial component of the electric vector in a slice of a sphere subjected to two plane waves coming from different directions. The time of observation is .5 times the period of the waves.

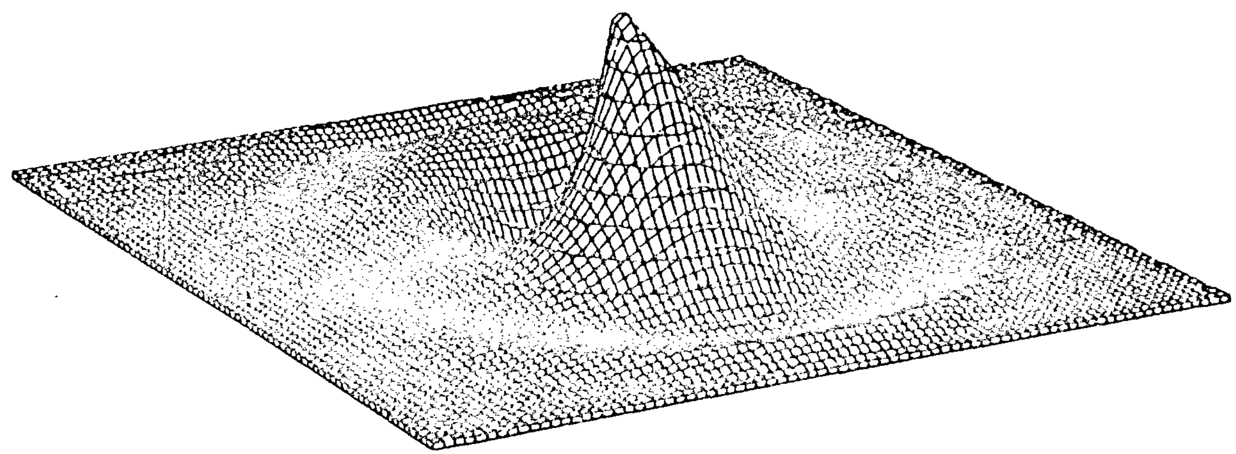
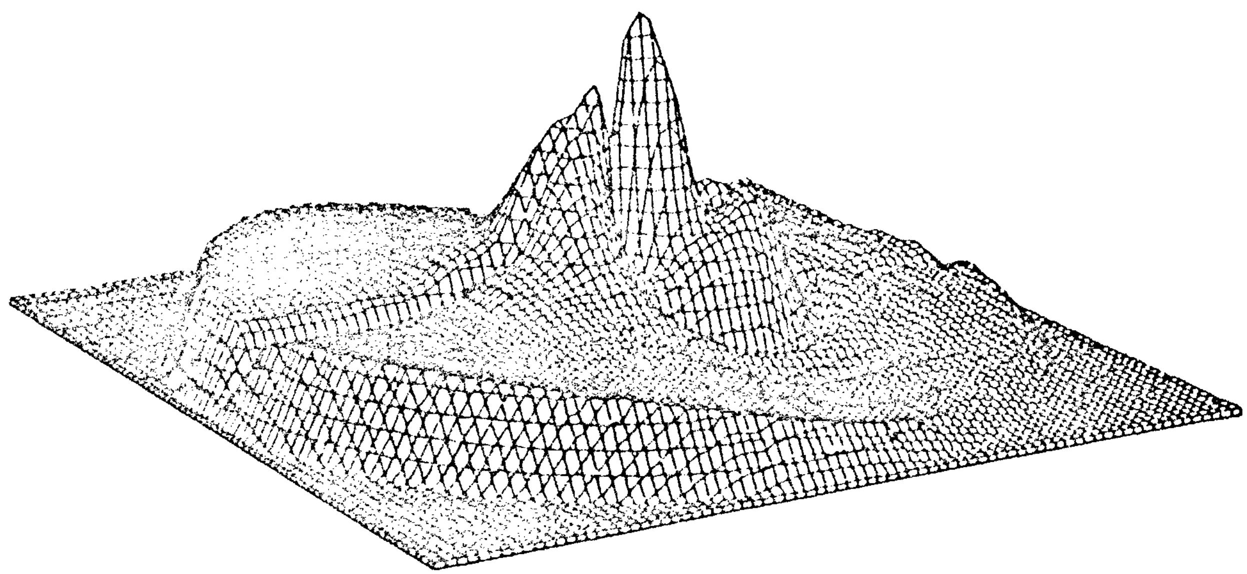


Figure 1.4.3. The figure below shows the length of the Poynting vector in a slice of the sphere subjected to plane waves of the same frequency coming from two different directions. All observation times give the same plot.



The program is capable of producing on any intersection of the sphere with any plane passing through its center the

- the real part of any component of the electric or magnetic vector (6 different 3D plots)
- the imaginary part of any component of the electric or magnetic vector (6 different 3D plots)
- the absolute value of the Poynting vector
- the absolute value of the radial component of the Poynting vector
- the square of the length of the electric vector
- the square of the length of the magnetic vector

It was found that there are considerable possibilities for cooperative interactions of phase related sources with this brain tissue sphere described in Bell ([10]).

Let us assume that the incoming radiation is a plane wave traveling in the direction

$$\vec{e}_{z_b} = sin(\theta_b)cos(\phi_b)\vec{e}_x + sin(\theta_b)sin(\phi_b)\vec{e}_y + cos(\theta_b)\vec{e}_z \tag{1.4.1}$$

Assume further that we look at field distributions in the intersection of the structure with a plane passing through the origin and that the normal vector to this plane is

$$\vec{e}_{z_o} = sin(\theta_o)cos(\phi_o)\vec{e}_x + sin(\theta_o)sin(\phi_o)\vec{e}_y + cos(\theta_o)\vec{e}_z \tag{1.4.2}$$

The unit vector in the direction of the y_b axis is defined as a constant c times the cross product of $\vec{e}_z$ and the unit vector in the beam direction or

$$\vec{e}_{y_b} = c(\vec{e}_z \times \vec{e}_{z_b}) \tag{1.4.3}$$

which implies that

$$\vec{e}_{y_b} = -sin(\phi_b)\vec{e}_x + cos(\phi_b)\vec{e}_y \tag{1.4.4}$$

Then there is only one choice for the unit vector in the direction of the positive x_b axis, namely

$$\vec{e}_{x_b} = \vec{e}_{y_b} \times \vec{e}_b =$$

$$cos(\theta_b)cos(\phi_b)\vec{e}_x + cos(\theta_b)sin(\phi_b)\vec{e}_y + (-sin(\theta_b))\vec{e}_z \tag{1.4.5}$$

We assume that the electric vector of the incoming wave is polarized along the x_b axis and that we would like to use the answer to a different plane wave interaction problem associated with the direction of propagation being the laboratory z axis and the direction of polarization being the laboratory x axis. In case,

$$\theta_b = \theta_o = 0 \tag{1.4.6}$$

we see that $\theta = 0$ and

$$\phi = \phi_b - \phi_o \tag{1.4.7}$$

This fact is a specialization of the general relation,

$$\vec{e}_{x_b} \cdot \vec{e}_{x_o} =$$

$$cos(\theta_b)cos(\theta_o)cos(\phi_b - \phi_o) + sin(\theta_b)sin(\theta_o) \tag{1.4.8}$$

The rest of the story is obtained by simply computing the angle between the z_b axis and the z_o axis. By computing another dot product we find that

$$cos(\theta) = sin(\theta_b)sin(\theta_o)cos(\phi_b)cos(\phi_o)+$$

$$sin(\theta_b)sin(\theta_o)sin(\phi_b)sin(\phi_o) + cos(\theta_b)cos(\theta_o) \tag{1.4.9}$$

By requiring that *theta* be the inverse cosine of the right side of equation (1.4.9) we can use the standard Mie solution for plane wave incidence to determine the field distributions in the plane whose normal is $\vec{e}_o$. We note that in case

$$\phi_o = \phi_b = 0$$

that equation (1.4.9) implies that

$$cos(\theta) = cos(\theta_b - \theta_o)$$

which is exactly the solution that one would expect.

1.5 The Full Tensor Solution

We assume that if $\vec{V}$ is a vector, then

$$curl(\vec{V}) = \frac{1}{r sin(\theta)}\left[\frac{\partial}{\partial\theta}(sin(\theta)V_\phi) - \frac{\partial V_\theta}{\partial\phi}\right]\vec{e}_r \ +$$

$$\frac{1}{r}\left[\frac{1}{sin(\theta)}\left(\frac{\partial V_r}{\partial\phi}\right) - \frac{\partial}{\partial r}(rV_\phi)\right]\vec{e}_\theta \ +$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial}{\partial\theta}V_r\right]\vec{e}_\phi \qquad (1.5.1)$$

We then find that if we define vector fields $\vec{A}$, $\vec{B}$, $\vec{C}$ by the rules

$$\vec{A} = F(r)\vec{A}_{(m,n)} \qquad (1.5.2)$$

$$\vec{B} = F(r)\vec{B}_{(m,n)} \qquad (1.5.3)$$

$$\vec{C} = F(r)\vec{C}_{(m,n)} \qquad (1.5.4)$$

that then

$$curl(\vec{A}) = n(n+1)\frac{F(r)}{r}\vec{C}_{(m,n)} \ +$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rF(r))\vec{B}_{(m,n)}(\theta,\phi) \qquad (1.5.5)$$

$$curl(\vec{C}) = \frac{F(r)}{r}\vec{A}_{(m,n)} \qquad (1.5.6)$$

and

$$curl(\vec{B}) = -\frac{1}{r}\frac{\partial}{\partial r}(rF(r)\vec{A}_{(m,n)} \qquad (1.5.7)$$

For each pair (m,n) of indices we seek a special solution of Maxwell's equations in the full tensor bianisotropic material of the form,

$$\vec{E} \ = \ \mathcal{A}(r)\vec{A}_{(m,n)} + \mathcal{B}(r)\vec{B}_{(m,n)} + \mathcal{C}(r)\vec{C}_{(m,n)} \qquad (1.5.8)$$

We now attempt to find combinations of the functions $\mathcal{A}(r)$, $\mathcal{B}(r)$, and $\mathcal{C}(r)$ which satisfy Maxwell's equations. The first Maxwell equation obtained by taking the curl of both sides of equation (1.5.8) is, making use of equations (1.5.5), (1.5.7), and (1.5.6), we see that

$$curl(\vec{E}) \ = \ = n(n+1)\frac{\mathcal{A}(r)}{r}\vec{C}_{(m,n)} \ +$$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)(r\mathcal{A}(r))\vec{B}_{(m,n)} +$$
$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)(r\mathcal{B}(r))\vec{A}_{(m,n)} \; + \; \frac{\mathcal{C}(r)}{r}\vec{A}_{(m,n)}$$
$$= \; - \; i\omega\overline{\overline{\mu}}\vec{H} \; - \; \overline{\overline{\alpha}}\vec{E} \tag{1.5.9}$$

Thus, in general we see from equation (1.5.9) that there are linear functions, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$, of several variables such that

$$\mathcal{F} = \mathcal{F}(\mathcal{A}(r), \frac{\mathcal{A}(r)}{r}, \mathcal{A}'(r), \mathcal{B}(r), \frac{\mathcal{B}(r)}{r}, \mathcal{B}'(r), \mathcal{C}(r)), \tag{1.5.10}$$

by which we mean that there are constants f_j with

$$j \in \{1, 2, 3, 4, 5, 6, 7\} \tag{1.5.11}$$

such that

$$\mathcal{F} = f_1\mathcal{A}(r) \; + \; f_2\frac{\mathcal{A}(r)}{r} \; + \; f_3\mathcal{A}'(r) \; +$$
$$f_4\mathcal{B}(r) \; + \; f_5\frac{\mathcal{B}(r)}{r} \; + \; f_6\mathcal{B}'(r) \; + \; f_7\mathcal{C}(r), \tag{1.5.12}$$
$$\mathcal{G} = \mathcal{G}(\mathcal{A}(r), \frac{\mathcal{A}(r)}{r}, \mathcal{A}'(r), \mathcal{B}(r), \frac{\mathcal{B}(r)}{r}, \mathcal{B}'(r), \mathcal{C}(r)), \tag{1.5.13}$$

and similarly

$$\mathcal{H} = \mathcal{H}(\mathcal{A}(r), \frac{\mathcal{A}(r)}{r}, \mathcal{A}'(r), \mathcal{B}(r), \frac{\mathcal{B}(r)}{r}, \mathcal{B}'(r), \mathcal{C}(r)), \tag{1.5.14}$$

so that the magnetic field is given by

$$\vec{H} \; = \; \mathcal{F}(r)\vec{A}_{(m,n)} + \mathcal{G}(r)\vec{B}_{(m,n)} + \mathcal{H}(r)\vec{C}_{(m,n)} \tag{1.5.15}$$

We now obtain the final Maxwell equation by taking the curl of both sides of equation (1.5.15), and from it equations for a four parameter family of vector valued radial functions needed to represent the general field as a linear combination of solutions of the form (1.5.8) in a full tensor bianisotropic material. In the traditional Mie solution (Mie [34]) the radial functions are spherical Bessel functions.

$$curl(\vec{H}) \; = \; n(n+1)\frac{\mathcal{F}(r)}{r}\vec{C}_{(m,n)} +$$
$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)(r\mathcal{F}(r))\vec{B}_{(m,n)} +$$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)(r\mathcal{G}(r))\vec{A}_{(m,n)} \;+\; \frac{\mathcal{H}(r)}{r}\vec{A}_{(m,n)}$$

$$= \;-\;(i\omega\bar{\bar{\epsilon}}+\bar{\bar{\sigma}})\vec{E} \;+\; \bar{\bar{\beta}}\vec{H} \tag{1.5.16}$$

In order to see the general form of the last Maxwell equation (1.5.16) note, for example, that

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)(r\mathcal{F}) = \frac{1}{r}\left(\frac{\partial}{\partial r}\right)\Big[f_1 r\mathcal{A} + f_2\mathcal{A} + f_3 r\mathcal{A}'(r) +$$

$$f_4\mathcal{B}(r) + f_5\frac{\mathcal{B}(r)}{r} + f_6\mathcal{B}'(r) + f_7\mathcal{C}(r)\Big] \tag{1.5.17}$$

Expanding the right side of equation (1.5.17) we find that

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)(r\mathcal{F}) = f_1\frac{\mathcal{A}(r)}{r} + f_1\mathcal{A}'(r) + f_2\frac{\mathcal{A}'(r)}{r}$$

$$+ f_3\frac{\mathcal{A}'(r)}{r} + f_3\mathcal{A}''(r) + f_4\frac{\mathcal{B}(r)}{r} + f_4\mathcal{B}'(r) + f_5\frac{\mathcal{B}'(r)}{r} +$$

$$f_6\frac{\mathcal{B}'(r)}{r} + f_6\mathcal{B}''(r) + f_7\frac{\mathcal{C}(r)}{r} + f_7\mathcal{C}'(r) \tag{1.5.18}$$

We can then see that the final form of the resulting system of equations in the radial functions is given by,

$$\mathcal{K}\vec{A}_{(m,n)} + \mathcal{L}\vec{B}_{(m,n)} + \mathcal{M}\vec{C}_{(m,n)} \;=\; \vec{0} \tag{1.5.19}$$

We get three ordinary differential equations in the unknown radial functions $\mathcal{A}(r)$, $\mathcal{B}(r)$, and $\mathcal{C}(r)$ that are used to represent the electric vector. Assuming that the terms in the tensors are such that we can eliminate the undifferentiated function $\mathcal{C}$ from the equation obtained by equating coefficients of $\vec{C}_{(m,n)}$ on both sides of equation (1.5.16) we get a system of two simultaneous second order differential equations in the radial functions $\mathcal{A}$ and $\mathcal{B}$. A solution is specified by giving values of $\mathcal{A}$, $\mathcal{B}$ and their first derivatives at a prescribed point R_p, where $r = R_p$ might represent the outer spherical boundary of the layer of interest. Thus, there are four indpendent solutions in each layer. By emulating the solution of the specific example, which follows, we see that the complete solution is obtained by using continuity of tangential components of $\vec{E}$ and $\vec{H}$ across the spherical boundaries separating regions of continuity of tensorial electromagnetic properties. A solution $\vec{E}$ and $\vec{H}$ of Maxwell's equations is then for each Fourier mode corresponding to

the index m and in the layer corresponding to index p a linear combination of the four solutions corresponding to

$$(\mathcal{A}(R_p), \mathcal{A}'(R_p), \mathcal{B}(R_p), \mathcal{B}'(R_p)) = (1,0,0,0), \tag{1.5.20}$$

$$(\mathcal{A}(R_p), \mathcal{A}'(R_p), \mathcal{B}(R_p), \mathcal{B}'(R_p)) = (0,1,0,0), \tag{1.5.21}$$

$$(\mathcal{A}(R_p), \mathcal{A}'(R_p), \mathcal{B}(R_p), \mathcal{B}'(R_p)) = (0,0,1,0), \tag{1.5.22}$$

and

$$(\mathcal{A}(R_p), \mathcal{A}'(R_p), \mathcal{B}(R_p), \mathcal{B}'(R_p)) = (0,0,0,1), \tag{1.5.23}$$

As it will turn out that these functions depend only on n and and p and not on m. Thus, replacing $\mathcal{A}$ by $\mathcal{A}^{(j)}_{(n,p)}$, where j runs over the indices 1 through 4 to denote the four independent solutions, we see that the general representation of the electric vector in the pth layer is given by

$$\begin{aligned}\vec{E} = \sum_{(m,n)\in\mathcal{I}} \Bigg\{ \sum_{j=1}^{4} a^{(p,j)}_{(m,n)} \Big[\mathcal{A}^{(j)}_{(n,p)}(r)\vec{A}_{(m,n)}(\theta,\phi) \; + \\ \mathcal{C}^{(j)}_{(n,p)}(r)\vec{C}_{(m,n)}(\theta,\phi) \; + \; \mathcal{B}^{(j)}_{(n,p)}(r)\vec{B}_{(m,n)}(\theta,\phi) \Big] \Bigg\}\end{aligned} \tag{1.5.24}$$

Note that the three functions $\mathcal{A}^{(j)}_{(n,p)}$ and $\mathcal{B}^{(j)}_{(n,p)}$ and $\mathcal{C}^{(j)}_{(n,p)}$ appearing in equation (1.5.24) are not independent, but the linear combination in the summand of equation (1.5.24) represents a vector valued solution of Maxwell's equations in the full tensor bianisotropic material. Using equation (1.5.15) we see that we can write the the magnetic field in the form

$$\begin{aligned}\vec{H} = \sum_{(m,n)\in\mathcal{I}} \Bigg\{ \sum_{j=1}^{4} a^{(p,j)}_{(m,n)} \Big[\mathcal{F}^{(j)}_{(n,p)}(r)\vec{A}_{(m,n)}(\theta,\phi) \; + \\ \mathcal{H}^{(j)}_{(n,p)}(r)\vec{C}_{(m,n)}(\theta,\phi) \; + \; \mathcal{G}^{(j)}_{(n,p)}(r)\vec{B}_{(m,n)}(\theta,\phi) \Big] \Bigg\}\end{aligned} \tag{1.5.25}$$

Using equations (1.5.24) and (1.5.25) and requiring continuity of tangential components of $\vec{E}$ and $\vec{H}$ across the boundary $r = R_p$ we can relate expansion coefficients in layer p to those in layer $p+1$ by four equations in four unknowns. Certainly, if we had a perfectly conducting core there would be no trouble in reducing the number of unknowns at the first spherical boundary by requiring that the tangential component of the electric vector vanish on this surface. For a penetrable core the matter is a little more delicate as one

must select a pair of indpendent solutions with at worst an integrable singularity at the origin.

With what we have developed and will develop we can describe scattering of a general source by a spherical core that is

- hollow,
- perfectly conducting,
- anisotropic with diagonal tensors having the theta and phi components equal, and
- bianisotropic and satisfying the conditions described in the following section,

surrounded by any number of spherically symmetric layers which are bianisotropic with tensors satisfying conditions in the Heritage of Gauss paper ([13]).

In the remainder of this paper we discuss an example with nontrivial values of $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ where the electric and magnetic fields can be represented using Bessel functions with complex index and argument.

1.6 A Specific Class of Examples

We give a simple exact Mie like solution for a class of bianisotropic N layer magnetic, penetrable spherically symmetric structures. We consider a special class of diagonal $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ coupling tensors with complex numbers α_r and β_r being their radial parts and with complex numbers α and β being their tangential components and assume some additional special relations between these. We shall use a modified complex propagation constant k which in each layer has a square given by

$$k^2 = \omega^2\mu\epsilon - i\omega\mu\sigma + \alpha\beta \tag{1.6.1}$$

For the propagation constant defined by equation (1.6.1) we seek a simple Mie like electric vector solution of both the Faraday Maxwell equation (1.2.1) and the Ampere Maxwell equation (1.2.2) which has the form,

$$\vec{E} = \sum_{(m,n)\in\mathcal{I}} \Big\{ a_{(m,n)} \mathcal{Z}_n^{(a)}(kr) \vec{A}_{(m,n)}(\theta,\phi) \; +$$

$$c_{(m,n)}\frac{Z_n^{(c)}(kr)}{kr}\vec{C}_{(m,n)}(\theta,\phi)\ +$$

$$\frac{b_{(m,n)}}{kr}\left(-\left(\frac{\partial}{\partial r}\right)\left(rZ_n^{(b)}(kr)\right)\right)\vec{B}_{(m,n)}(\theta,\phi)\Bigg\} \tag{1.6.2}$$

where the three radial functions $Z_n^{(a)}$, $Z_n^{(b)}$, and $Z_n^{(c)}$ are to be derived and the functions $\vec{A}_{(m,n)}$, $\vec{B}_{(m,n)}$, and $\vec{C}_{(m,n)}$ are defined by equations (1.3.1), (1.3.2), and (1.3.3). We shall derive relationships needed between the radial functions $Z_n^{(a)}$, $Z_n^{(b)}$, and $\vec{C}_{(m,n)}$ and the complex expansion coefficients $a_{(m,n)}$, $b_{(m,n)}$, and $c_{(m,n)}$ needed to get an interesting, but easily computed, exact Mie like solutions for a the reponse of a class of N layer bianisotropic spheres to both plane waves and complex sources.

We now begin to develop the consequences of Maxwell's equations by noting that equation (1.6.2) and the three basic curl relationships, (1.5.5), (1.5.6), and (1.5.7) and the Faraday Maxwell equation (1.2.1) imply that $curl(\vec{E})$ is given by

$$curl(\vec{E}) =$$

$$\sum_{(m,n)\in\mathcal{I}}\Bigg\{a_{(m,n)}\left[n(n+1)\frac{Z_n^{(a)}(kr)}{r}\vec{C}_{(m,n)}\ +\ \frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right)\vec{B}_{(m,n)}\right]\ +$$

$$c_{(m,n)}\frac{Z_n^{(c)}(kr)}{kr^2}\vec{A}_{(m,n)}\ +\ b_{(m,n)}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2(rZ_n^{(b)}(kr))\vec{A}_{(m,n)}\Bigg\} =$$

$$-\overline{\overline{\alpha}}\left[\sum_{(m,n)\in\mathcal{I}}\Bigg\{a_{(m,n)}Z_n^{(a)}(kr)\vec{A}_{(m,n)}\ +\ c_{(m,n)}\frac{Z_n^{(c)}}{kr}\vec{C}_{(m,n)}\ +\right.$$

$$\left.\frac{b_{(m,n)}}{kr}\left(-\left(\frac{\partial}{\partial r}\right)\left(rZ_n^{(b)}(kr)\right)\right)\vec{B}_{(m,n)}\Bigg\}\right] - i\omega\overline{\overline{\mu}}\vec{H} \tag{1.6.3}$$

This is the completely general Faraday Maxwell equation for electric vectors given by equation (1.6.2). We want to solve equation (1.6.3) for $\vec{H}$ so that we can substitute this vector valued function into the Ampere Maxwell equation (1.2.2) and determine what types of equations the three radial functions $Z_n^{(a)}$, $Z_n^{(b)}$, and $Z_n^{(c)}$ should satisfy. Solving equation (1.6.3) for $\overline{\overline{\mu}}\vec{H}$, we see that in general if we simply assume that $\overline{\overline{\alpha}}$ is a diagonal tensor whose action on a vector represented in spherical coordinates is defined by,

$$\overline{\overline{\alpha}}\cdot\vec{E} = \begin{pmatrix}\alpha_r & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{pmatrix}\begin{pmatrix}E_r\\ E_\theta\\ E_\phi\end{pmatrix} \tag{1.6.4}$$

that then making use of equation (1.6.4)

$$-i\omega\overline{\overline{\mu}}\vec{H} =$$

$$\sum_{(m,n)\in\mathcal{I}} \left\{ \left[a_{(m,n)} n(n+1) \frac{Z_n^{(a)}(kr)}{r} + \alpha_r c_{(m,n)} \frac{Z_n^{(c)}(kr)}{kr} \right] \vec{C}_{(m,n)} \right.$$

$$+ \left[a_{(m,n)} \frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right) - \alpha b_{(m,n)} \left(\frac{1}{kr}\right)\left(\frac{\partial}{\partial r}\right)\left(rZ_n^{(b)}(kr)\right) \right] \vec{B}_{(m,n)} +$$

$$\left. \left[c_{(m,n)} \frac{Z_n^{(c)}(kr)}{kr^2} + b_{(m,n)} \frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2 (rZ_n^{(b)}(kr)) + \alpha a_{(m,n)} Z_n^{(a)}(kr) \right] \vec{A}_{(m,n)} \right\} \quad (1.6.5)$$

In the previous section we allowed $\overline{\overline{\alpha}}$, $\overline{\overline{\beta}}$, $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$, and $\overline{\overline{\sigma}}$ to be general tensors and solved for the general radial functions $\mathcal{C}(r)$, $\mathcal{B}(r)$, and $\mathcal{A}(r)$ used in the represenatation (1.5.8) of the electric vector, we now assume that all these tensors have the same form as the complex $\overline{\overline{\alpha}}$ tensor given by equation (1.6.4).

With these assumptions, we see that for our simplified bianisotropic material, the magnetic vector will then have the form

$$\vec{H} =$$

$$\sum_{(m,n)\in\mathcal{I}} \left[\frac{i}{\omega\mu_r} \left\{ a_{(m,n)} \frac{Z_n^{(a)}(kr)n(n+1)}{r} + \alpha_r c_{(m,n)} \frac{Z_n^{(c)}(kr)}{kr} \right\} \vec{C}_{(m,n)} \right.$$

$$+ \frac{i}{\omega\mu} \left\{ a_{(m,n)} \frac{1}{r}\frac{\partial}{\partial r}(rZ_n^{(a)}(kr) - \alpha b_{(m,n)} \frac{1}{kr}\left(\frac{\partial}{\partial r}\right)\left(rZ_n^{(b)}(kr)\right) \right\} \vec{B}_{(m,n)} +$$

$$\left. \frac{i}{\omega\mu} \left\{ c_{(m,n)} \frac{Z_n^{(c)}(kr)}{kr^2} + b_{(m,n)} \left(\frac{1}{kr}\right)\left(\frac{\partial}{\partial r}\right)^2 \left(rZ_n^{(b)}(kr)\right) + \alpha a_{(m,n)} Z_n^{(a)}(kr) \right\} \vec{A}_{(m,n)} \right] \quad (1.6.6)$$

Applying the *curl* operation to both sides of equation 1.6.6 using the three curl equations (1.5.5), (1.5.6) and (1.5.7) we obtain an expanded form of the Ampere Maxwell equation given by,

$$curl(\vec{H}) = \left(\frac{i}{\omega\mu_r}\right) \frac{1}{r} \left\{ a_{(m,n)} \frac{Z_n^{(a)}(kr)n(n+1)}{r} + \alpha_r c_{(m,n)} \frac{Z_n^{(c)}(r)}{kr} \right\} \vec{A}_{(m,n)} +$$

$$\left(\frac{i}{\omega\mu}\right)\left(\frac{-1}{r}\frac{\partial}{\partial r}\right)\left(r\left\{a_{(m,n)}\frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right)\right.\right.$$

$$\left.\left. -\alpha b_{(m,n)} \frac{1}{kr}\frac{\partial}{\partial r}\left(rZ_n^{(b)}(kr)\right)\right)\right\} \vec{A}_{(m,n)} +$$

$$\left(\frac{i}{\omega\mu}\right)\frac{n(n+1)}{r}\left\{c_{(m,n)}\frac{Z_n^{(c)}(kr)}{kr^2}\right. +$$

$$\left. b_{(m,n)}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(kr)\right) + \alpha a_{(m,n)}Z_n^{(a)}(kr)\right\}\vec{C}_{(m,n)} +$$

$$\left(\frac{i}{\omega\mu}\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r\left\{c_{(m,n)}\frac{Z_n^{(c)}(kr)}{kr^2}\right.\right. +$$

$$\left.\left. b_{(m,n)}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(kr)\right) + \alpha a_{(m,n)}Z_n^{(a)}(kr)\right\}\right)\vec{B}_{(m,n)}$$

$$= (i\omega\bar{\bar{\epsilon}} + \bar{\bar{\sigma}})\vec{E} + \bar{\bar{\beta}}\vec{H} \tag{1.6.7}$$

To make use of the Ampere Maxwell equation (1.6.7), we need to use our original equation (1.6.2) for $\vec{E}$ and equation (1.6.6) for $\vec{H}$ to obtain

$$(i\omega\bar{\bar{\epsilon}} + \bar{\bar{\sigma}})\vec{E} + \bar{\bar{\beta}}\vec{H} =$$

$$\sum_{(m,n)\in\mathcal{I}}\left\{\left[(i\omega\epsilon+\sigma)a_{(m,n)}Z_n^{(a)}(r) + \frac{i\beta}{\omega\mu}\left\{\frac{c_{(m,n)}}{kr^2}Z_n^{(c)}(kr)\right.\right.\right. +$$

$$\left.\left. \frac{b_{(m,n)}}{kr}\left(\frac{\partial}{\partial r}\right)^2(rZ_n^{(b)}(kr)) + \alpha a_{(m,n)}Z_n^{(a)}(kr)\right\}\right]\vec{A}_{(m,n)}(\theta,\phi) +$$

$$\left[(i\omega\epsilon+\sigma)\frac{b_{(m,n)}}{kr}\left(-\frac{\partial}{\partial r}\right)\left(rZ_n^{(b)}(kr)\right)\right. +$$

$$\left.\left[\frac{i\beta}{\omega\mu}\left\{a_{(m,n)}\frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right) + \frac{\alpha b_{(m,n)}}{kr}\left(-\frac{\partial}{\partial r}(rZ_n^{(b)}(kr))\right)\right\}\right]\right]\vec{B}_{(m,n)}(\theta,\phi)$$

$$+\left[(i\omega\epsilon_r+\sigma_r)c_{(m,n)}\frac{Z_n^{(c)}(kr)}{kr}\right. +$$

$$\left.\frac{i\beta_r}{\omega\mu_r}\left\{a_{(m,n)}Z_n^{(a)}(kr)\frac{n(n+1)}{r} + \alpha_r\frac{c_{(m,n)}Z_n^{(c)}(r)}{kr}\right\}\right]\vec{C}_{(m,n)}(\theta,\phi) \tag{1.6.8}$$

The solution of the electromagnetic interaction problem is then obtained by relating coefficients on both sides of equation (1.6.7) and making use of orthogonality relations to get differential equations for the, a priori unknown, radial functions, $Z_n^{(a)}$, $Z_n^{(b)}$, and $Z_n^{(c)}$.

Equation (1.6.7) coupled with equation (1.6.8) is the key to the development of a system of ordinary differential equations satisfied by the radial functions. Using orthogonality properties of the vector functions $\vec{A}_{(m,n)}$ and $\vec{B}_{(m,n)}$ and $\vec{C}_{(m,n)}$ defined, respectively, by (1.3.1) and (1.3.2), and (1.3.3) we shall develop three relationships involving only the radial

functions, express one of these radial functions in terms of the others, and get an uncoupled system whose solutions will be Bessel functions with complex index and argument. by equating their coefficients on both sides of equation (1.6.7). Equating coefficients of $\vec{A}_{(m,n)}$ on both sides of equation (1.6.7) we find that

$$\left[\left(\frac{i}{\omega\mu_r}\right)\frac{1}{r}\left\{a_{(m,n)}\frac{Z_n^{(a)}(kr)n(n+1)}{r}+\alpha_r c_{(m,n)}\frac{Z_n^{(c)}(kr)}{kr}\right\}+\right.$$
$$\left(\frac{i}{\omega\mu}\right)\left(\frac{-1}{r}\frac{\partial}{\partial r}\right)\left(r\left\{a_{(m,n)}\frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right)\right.\right.$$
$$\left.\left.\left.-\alpha b_{(m,n)}\frac{1}{kr}\frac{\partial}{\partial r}\left(rZ_n^{(b)}(kr)\right)\right)\right\}\right]=$$
$$\left[(i\omega\epsilon+\sigma)a_{(m,n)}Z_n^{(a)}(kr)+\frac{i\beta}{\omega\mu}\left\{\frac{c_{(m,n)}}{kr^2}Z_n^{(c)}(kr)+\right.\right.$$
$$\left.\left.\frac{b_{(m,n)}}{kr}\left(\frac{\partial}{\partial r}\right)^2(rZ_n^{(b)}(kr))+\alpha a_{(m,n)}Z_n^{(a)}(kr)\right\}\right] \tag{1.6.9}$$

We can see the consistency of this equation with the equations obtained for the special case of anisotropic spherical structures ([15]). If in equation (1.6.9) we set $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ equal to the zero tensor, we obtain

$$\left[\left(\frac{i}{\omega\mu_r}\right)\frac{1}{r}\left\{a_{(m,n)}\frac{Z_n^{(a)}(kr)n(n+1)}{r}\right\}+\right.$$
$$\left.\left(\frac{i}{\omega\mu}\right)\left(\frac{-1}{r}\frac{\partial}{\partial r}\right)\left(r\left\{a_{(m,n)}\frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right)\right\}\right]=\left[(i\omega\epsilon+\sigma)a_{(m,n)}Z_n^{(a)}(kr)\right] \tag{1.6.10}$$

or upon multiplying both sides of equation (1.6.10) by $-i\omega\mu$ we find that if we define the propagation constant for a class of anisotropic structures ([15]) by the rule,

$$k_a^2=\omega^2\mu\epsilon-i\omega\mu\sigma \tag{1.6.11}$$

that then $Z_n^{(a)}$ satisfies,

$$\left(\frac{\mu}{\mu_r}\right)Z_n^{(a)}(k_ar)\frac{n(n+1)}{r^2}-k^2Z_n^{(a)}(k_ar)=\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2(rZ_n^{(a)}(k_ar)) \tag{1.6.12}$$

which, with the propagation constant k_a being defined by (1.6.11) rather than by (1.6.1), is exactly the equation satisfied by the radial function $Z_n^{(a)}$ for an anisotropic sphere ([15]).

We can also, in a similar fashion, relate coefficients of $\vec{B}_{(m,n)}$ on both sides of equation (1.6.7) with k^2 defined by (1.6.1) to obtain the relationship.

$$\left(\frac{i}{\omega\mu}\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r\left\{c_{(m,n)}\frac{Z_n^{(c)}(kr)}{kr^2}+\right.\right.$$

$$\left.\left. b_{(m,n)}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(kr)\right)+\alpha a_{(m,n)}Z_n^{(a)}(kr)\right\}\right)=$$

$$\left[(i\omega\epsilon+\sigma)\frac{b_{(m,n)}}{kr}\left(-\frac{\partial}{\partial r}\right)\left(rZ_n^{(b)}(kr)\right)+\right.$$

$$\left[\frac{i\beta}{\omega\mu}\left\{a_{(m,n)}\frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right)+\frac{\alpha b_{(m,n)}}{kr}\left(-\frac{\partial}{\partial r}(rZ_n^{(b)}(kr))\right)\right\}\right]\Bigg] \qquad (1.6.13)$$

If in equation (1.6.13) we equate the terms operated on by 1 over r times the partial derivative with respect to r, and then divide all terms on both sides by r we deduce that equation (1.6.13) is implied by the simpler relation,

$$\left(\frac{i}{\omega\mu}\right)\left\{c_{(m,n)}\frac{Z_n^{(c)}(kr)}{kr^2}\quad+\right.$$

$$\left. b_{(m,n)}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(r)\right)+\alpha a_{(m,n)}Z_n^{(a)}(r)\right\}\Bigg)=$$

$$\left[(i\omega\epsilon+\sigma)\frac{b_{(m,n)}}{k}\left(-Z_n^{(b)}(r)\right)\cdot+\right.$$

$$\left[\frac{i\beta}{\omega\mu}\left\{a_{(m,n)}\left(Z_n^{(a)}(r)\right)-\frac{\alpha b_{(m,n)}}{k}\left(\left(Z_n^{(b)}(r)\right)\right)\right\}\right]\Bigg] \qquad (1.6.14)$$

Equating coefficients of $\vec{C}_{(m,n)}$ on both sides of our specialized Ampere Maxwell equation (1.6.7) with k defined by (1.6.1) reveals that

$$\left(\frac{i}{\omega\mu}\right)\frac{n(n+1)}{r}\left\{c_{(m,n)}\frac{Z_n^{(c)}(kr)}{kr^2}\quad+\right.$$

$$\left. b_{(m,n)}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(kr)\right)+\alpha a_{(m,n)}Z_n^{(a)}(kr)\right\}=$$

$$+\left[(i\omega\epsilon_r+\sigma_r)c_{(m,n)}\frac{Z_n^{(c)}(kr)}{kr}\quad+\right.$$

$$\left.\frac{i\beta_r}{\omega\mu_r}\left\{a_{(m,n)}Z_n^{(a)}(kr)\frac{n(n+1)}{r}+\alpha_r\frac{c_{(m,n)}Z_n^{(c)}(kr)}{kr}\right\}\right] \qquad (1.6.15)$$

To compare equation (1.6.14) and equation (1.6.15) we multiply both sides of equation (1.6.15) by $r/(n(n+1))$ and we find that

$$\left(\frac{i}{\omega\mu}\right)\left\{\frac{c_{(m,n)}}{kr^2}Z_n^{(c)}(kr)+b_{(m,n)}\left(\frac{1}{kr}\right)\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(kr)\right)+\alpha a_{(m,n)}Z_n^{(a)}(kr)\right\}=$$

$$\left(\frac{i\omega\epsilon_r+\sigma_r}{n(n+1)}\right)c_{(m,n)}\frac{Z_n^{(c)}(kr)}{k}\;+$$

$$\beta_r\frac{i}{\omega\mu_r}a_{(m,n)}Z_n^{(a)}(kr)+\left(\frac{\alpha_r\beta_r}{n(n+1)}\right)\left(\frac{i}{\omega\mu_r}c_{(m,n)}\right)\frac{Z_n^{(c)}(kr)}{k} \tag{1.6.16}$$

Since the left side of equation (1.6.16) is identical to the left side of equation (1.6.14) it is clear that we have consistency between equation (1.6.16) and equation (1.6.14) provided that

$$(i\omega\epsilon+\sigma)(-b_{(m,n)}Z_n^{(b)}(r))\;+$$

$$\left(\frac{i\beta}{\omega\mu}\right)ka_{m,n}Z_n^{(a)}(kr)-\left(\frac{i\beta\alpha}{\omega\mu}\right)b_{(m,n)}Z_n^{(b)}(kr)=$$

$$\left(\frac{i\omega\epsilon_r+\sigma_r}{n(n+1)}\right)c_{(m,n)}Z_n^{(c)}(kr)\;+$$

$$\frac{i\beta_r k}{\omega\mu_r}a_{(m,n)}Z_n^{(a)}(kr)+\frac{\alpha_r\beta_r}{n(n+1)}\left(\frac{i}{\omega\mu_r}\right)c_{(m,n)}Z_n^{(n)}(kr) \tag{1.6.17}$$

where k is defined by (1.6.1).

We note that the consistency relation given by equation (1.6.17) specializes for the case of the ordinary anisotropic sphere, where the coupling tensors $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$, are both equal to the zero tensor, to the simple anisotropic sphere relation of ([15]) given by

$$(i\omega\epsilon+\sigma)(-b_{(m,n)})Z_n^{(b)}(k_ar)=\left(\frac{i\omega\epsilon_r+\sigma_r}{n(n+1)}\right)c_{(m,n)}Z_n^{(c)}(k_ar) \tag{1.6.18}$$

where k_a is given by (1.6.11) We note that equation (1.6.18) is satisfied if

$$Z_n^{(c)}(k_ar)=Z_n^{(b)}(k_ar) \tag{1.6.19}$$

and

$$c_{(m,n)}=-n(n+1)\left(\frac{i\omega\epsilon+\sigma}{i\omega\epsilon_r+\sigma_r}\right)b_{(m,n)} \tag{1.6.20}$$

which are identical to the relations derived in ([15]) for anisotropic spheres.

We, however, now again suppose that k is defined more generally by (1.6.1) and collect the terms multiplying the coefficients $a_{(m,n)}$, $b_{(m,n)}$, and $c_{(m,n)}$ in equation (1.6.9). In doing so we rewrite (1.6.9) in the form,

$$\left[\left(\frac{i}{\omega\mu_r}\right)\left(\frac{1}{r}\right) Z_n^{(a)}(kr)\frac{n(n+1)}{r} - (i\omega\epsilon+\sigma)Z_n^{(a)}(kr)\right.$$

$$\left. -\frac{i\beta}{\omega\mu}\alpha Z_n^{(a)}(kr) - \frac{i}{\omega\mu}\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2 (rZ_n^{(a)}(kr)\right] a_{(m,n)} \; +$$

$$\left[\left(\frac{-i}{\omega\mu}\right)\frac{\alpha}{k} - \frac{i\beta}{\omega\mu k}\right]\left\{\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2 (rZ_n^{(b)}(kr))\right\} b_{(m,n)} \; +$$

$$\left[\left(\frac{i\alpha_r}{\omega\mu_r} - \frac{i\beta}{\omega\mu}\frac{1}{kr^2}\right) Z_n^{(c)}(kr)\right] c_{(m,n)} = 0 \qquad (1.6.21)$$

By rearranging equations as we have done we are attempting to develop, for our class of bianisotropic spheres, relationships for the radial functions analogous to the relationships, (1.6.12) and (1.6.19), for anisotropic spheres. Thus, collecting the coefficients of $a_{(m,n)}$, $b_{(m,n)}$, and $c_{(m,n)}$ in equation (1.6.14) we have

$$\left[\left(\frac{i}{\omega\mu}\right)\alpha Z_n^{(a)}(kr) - \frac{i\beta}{\omega\mu}Z_n^{(a)}(kr)\right] a_{(m,n)}$$

$$+\left[\frac{i}{\omega\mu}\left(\frac{1}{kr}\right)\left(\frac{\partial}{\partial r}\right)^2 (rZ_n^{(b)}(kr)) + \frac{\sigma+i\omega\epsilon}{k}Z_n^{(b)}(kr)\right.$$

$$\left. + \frac{i\beta\alpha}{\omega\mu k}Z_n^{(b)}(kr)\right] b_{(m,n)} + \frac{i}{\omega\mu}\left(\frac{1}{kr^2}\right) Z_n^{(c)}(kr)c_{(m,n)} = 0 \qquad (1.6.22)$$

Equation (1.6.22) yields the relationship

$$b_{(m,n)}\frac{i}{\omega\mu}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2 (rZ_n^{(b)}(kr) \; =$$

$$-\left[\frac{i}{\omega\mu}\alpha Z_n^{(a)}(kr) - \frac{i\beta}{\omega\mu}Z_n^{(a)}(kr)\right] a_{(m,n)} \; -$$

$$\left[\frac{\sigma+i\omega\epsilon}{k}Z_n^{(b)}(kr) \; + \; \frac{i\beta\alpha}{\omega\mu k}Z_n^{(b)}(kr)\right] b_{(m,n)}$$

$$-\frac{i}{\omega\mu}\frac{1}{kr^2}Z_n^{(c)}(kr)c_{(m,n)} \qquad (1.6.23)$$

with the k being given by (1.6.1).

Equation (1.6.15), after collecting terms multiplying the same coefficients, yields the relationship

$$\frac{n(n+1)}{r}\left\{b_{(m,n)}\frac{i}{\omega\mu}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2 (rZ_n^{(b)}(kr))\right\}$$

$$= \left\{\frac{-n(n+1)}{r}\left(\frac{i}{\omega\mu}\frac{1}{kr^2}Z_n^{(c)}(kr)\right) + \beta_r\left(\frac{i}{\omega\mu_r}\right)\frac{\alpha_r}{kr}Z_n^{(c)}(kr)\right.$$

$$\left. + \frac{i\omega\epsilon_r + \sigma_r}{kr}Z_n^{(c)}(kr)\right\} c_{(m,n)} +$$

$$a_{(m,n)}\left\{\frac{-i}{\omega\mu}\frac{n(n+1)}{r}\alpha Z_n^{(a)}(kr) + \beta_r\left(\frac{i}{\omega\mu_r}\right)Z_n^{(a)}(kr)\frac{n(n+1)}{r}\right\} \tag{1.6.24}$$

Multiplying all terms of equation (1.6.24) by $r/(n(n+1))$ we find that

$$\left\{b_{(m,n)}\frac{i}{\omega\mu}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2 (rZ_n^{(b)}(kr))\right\} =$$

$$\left\{-\left[\frac{i}{\omega\mu}\frac{1}{kr^2}Z_n^{(c)}(r)\right] + \left(\frac{i\alpha_r\beta_r}{\omega\mu_r}\right)\frac{1}{kn(n+1)}Z_n^{(c)}(kr)\right.$$

$$\left. + \frac{i\omega\epsilon_r + \sigma_r}{kn(n+1)}Z_n^{(c)}(kr)\right\} c_{(m,n)} +$$

$$\left\{\frac{-i}{\omega\mu}\alpha Z_n^{(a)}(kr) + \left(\frac{i\beta_r}{\omega\mu_r}\right)Z_n^{(a)}(kr)\right\} a_{(m,n)} \tag{1.6.25}$$

Solving for the term

$$U = \frac{i}{\omega\mu}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2 (rZ_n^{(b)}(kr))b_{(m,n)} \tag{1.6.26}$$

in equations (1.6.25) and (1.6.23) we find that equating the two expressions for U yields,

$$-\left[\alpha\frac{i}{\omega\mu}Z_n^{(a)}(kr) - \frac{i\beta}{\omega\mu}Z_n^{(a)}(kr)\right] a_{(m,n)}$$

$$-\left[\frac{\sigma + i\omega\epsilon}{k}Z_n^{(b)}(kr) + \frac{i\beta}{\omega\mu}\frac{\alpha}{k}Z_n^{(b)}(kr)\right] b_{(m,n)}$$

$$-c_{(m,n)}\left(\frac{i}{kr^2\omega\mu}\right)Z_n^{(c)}(kr) =$$

$$-\left[\frac{i\alpha}{\omega\mu}Z_n^{(a)}(kr) - \frac{i\beta_r}{\omega\mu_r}Z_n^{(a)}(kr)\right] a_{(m,n)} -$$

$$\left[\left(\frac{i}{\omega\mu}\right)\frac{1}{kr^2}Z_n^{(c)}(kr) - \frac{i\omega\epsilon_r + \sigma_r}{kn(n+1)}Z_n^{(c)}(kr)\right.$$

$$-\left(\frac{\alpha_r\beta_r}{k\omega\mu_r n(n+1)}\right) Z_n^{(c)}(kr)\Big] c_{(m,n)} \tag{1.6.27}$$

Equation (1.6.27) implies, after subtracting identical terms from both sides of the equation, that

$$\left[-\frac{i\omega\epsilon_r+\sigma_r}{kn(n+1)}-\frac{\beta_r\alpha_r}{k\omega\mu_r n(n+1)}\right] c_{(m,n)} Z_n^{(c)}(kr) =$$

$$-\left(\frac{i\beta}{\omega\mu}-\frac{i\beta_r}{\omega\mu_r}\right) Z_n^{(a)}(kr) a_{(m,n)} \ +$$

$$\left[\frac{\sigma+i\omega\epsilon}{k}+\frac{i\beta\alpha}{\omega\mu k}\right] Z_n^{(b)}(kr) b_{(m,n)} \tag{1.6.28}$$

Solving equation (1.6.17) for $c_{(m,n)}Z_n^{(c)}(kr)$ we find that

$$\left\{\frac{i\omega\epsilon_r+\sigma_r}{n(n+1)}+\frac{i\alpha_r\beta_r}{\omega\mu_r n(n+1)}\right\} c_{(m,n)} Z_n^{(c)}(kr) \ =$$

$$\left\{-(i\omega\epsilon+\sigma)-\frac{i\beta\alpha}{\omega\mu}\right\} b_{(m,n)} Z_n^{(b)}(kr) \ +$$

$$\left(\frac{i\beta k}{\omega\mu}-\frac{i\beta_r k}{\omega\mu_r}\right) a_{(m,n)} Z_n^{(a)}(kr) \tag{1.6.29}$$

We could use this relationship (1.6.29) to eliminate $Z_n^{(c)}$ but we would end up with a coupled system in the other two radial functions. However, for a simpler chiral sphere where

$$\frac{\beta}{\mu}=\frac{\beta_r}{\mu_r}, \tag{1.6.30}$$

equation (1.6.29) has the form

$$c_{(m,n)} Z_n^{(c)}(kr) = -n(n+1)\left(\frac{i\omega\epsilon+\sigma+i\beta\alpha/(\omega\mu)}{(i\omega\epsilon_r+\sigma_r)+(i\alpha_r\beta_r)/(\omega\mu_r)}\right) b_{(m,n)} Z_n^{(b)}(kr) \tag{1.6.31}$$

If we assume that equation (1.6.29) is satisfied, and equation (1.6.30) is valid so that equation (1.6.31) is valid and, furthermore, that

$$\frac{i\alpha}{\omega\mu}=\frac{i\beta}{\omega\mu}, \tag{1.6.32}$$

then equation (1.6.14) will be of the form

$$\frac{i}{\omega\mu}\left[-n(n+1)\left(\frac{i\omega\epsilon+\sigma+i\beta\alpha/(\omega\mu)}{(i\omega\epsilon_r+\sigma_r)+(i\alpha_r\beta_r)/(\omega\mu_r)}\right)\right] b_{(m,n)} \frac{Z_n^{(b)}(r)}{kr^2}$$

$$\left(\frac{i}{\omega\mu}\right) b_{(m,n)} \left(\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2\right) (rZ_n^{(b)}(r)) =$$

$$\frac{\sigma + i\omega\epsilon}{k} b_{(m,n)}(-Z_n^{(b)}(r)) - \frac{i\beta\alpha}{\omega\mu k} b_{(m,n)} Z_n^{(b)}(r) \qquad (1.6.33)$$

If we also impose the condition

$$\frac{i\alpha_r}{\omega\mu_r} = \frac{i\beta}{\omega\mu} \qquad (1.6.34)$$

then equation (1.6.9) takes on the form

$$\left(\frac{in(n+1)}{\omega\mu_r r^2}\right) Z_n^{(a)}(r) + \left(\frac{-i}{\omega\mu}\right) \frac{1}{r} \left(\frac{\partial}{\partial r}\right)^2 (r Z_n^{(a)}(r)) =$$

$$(i\omega\mu + \sigma) Z_n^{(a)}(r) + \frac{i\beta\alpha}{\omega\mu} Z_n^{(a)} \qquad (1.6.35)$$

Multiplying all terms of equation (1.6.35) by $i\omega\mu$ and observing that

$$k^2 = k_a^2 + \alpha \cdot \beta \qquad (1.6.36)$$

where k_a is defined by (1.6.11) and k is defined by (1.6.1) we see that

$$-\frac{\mu n(n+1)}{\mu_r r^2} Z_n^{(a)} + \frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2 (r Z_n^{(a)}(r)) =$$

$$-k_a^2 Z_n^{(a)}(r) - \alpha\beta Z_n^{(a)}(r) \qquad (1.6.37)$$

or if we introduce the variable

$$\zeta_a = \frac{\mu}{\mu_r} \qquad (1.6.38)$$

the ordinary differential equation (1.6.37) satisfied by $Z_n^{(a)}(kr)$ is

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2 (r Z_n^{(a)}(kr)) + \left[(k_a^2 + \alpha\beta) - \zeta_a \frac{n(n+1)}{r^2}\right] Z_n^{(a)}(kr) = 0 \qquad (1.6.39)$$

where k_a is given by (1.6.11) and k is given by (1.6.1)

The spherical Bessel function is defined as

$$\Psi_\nu(z) = \frac{\sqrt{\pi} J_{\nu+1/2}(z)}{\sqrt{2}\sqrt{z}} \qquad (1.6.40)$$

where $\Psi_\nu(z)$ satisfies

$$\frac{1}{z}\left(\frac{\partial}{\partial z}\right)^2 (z\Psi_\nu(z)) + \left[1 + \frac{\nu(\nu+1)}{z^4}\right] \Psi_\nu(z) = 0 \qquad (1.6.41)$$

Dividing all terms of equation (1.6.37) by

$$k^2 = k_a^2 + \alpha\beta = \omega^2\mu\epsilon + \beta\alpha - i\omega\mu\sigma \tag{1.6.42}$$

we have with the definition

$$z^2 = (\omega^2\mu\epsilon - i\omega\mu\sigma + \beta\alpha)r^2 = k^2r^2 \tag{1.6.43}$$

the fact that equation (1.6.39) implies

$$\frac{1}{z}\left(\frac{\partial}{\partial z}\right)^2 (z\Psi_\nu(z)) + \left[1 - \frac{\zeta_a n(n+1)}{z^2}\right]\Psi_\nu = 0 \tag{1.6.44}$$

where

$$\nu(\nu+1) = \zeta_a n(n+1) \tag{1.6.45}$$

We can find a simple formula for the index ν of the form

$$\nu = \frac{-1 + \sqrt{1 + 4\zeta n(n+1)}}{2} \tag{1.6.46}$$

Equation (1.6.33) gives the second equation which implies that

$$\frac{i}{\omega\mu}\left\{\frac{1}{kr^2}\right\}\left[-n(n+1)\left(\frac{i\omega\epsilon + \sigma + i\beta\alpha/(\omega\mu)}{(i\omega\epsilon_r + \sigma_r) + (i\alpha_r\beta_r)/(\omega\mu_r)}\right)\right] Z_n^{(b)}(kr)$$

$$\left(\frac{i}{\omega\mu}\right)\left(\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2\right)(rZ_n^{(b)}(kr)) =$$

$$\frac{\sigma + i\omega\epsilon}{k}(-Z_n^{(b)}(kr)) - \frac{i\beta\alpha}{\omega\mu k}Z_n^{(b)}(kr) \tag{1.6.47}$$

Multiplying all terms of equation (1.6.47) by $-i\omega\mu kr^2$ and using equation (1.6.36) we deduce from equation (1.6.47) that

$$\left(\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\right)(rZ_n^{(b)}(kr)) + (k_a^2 + \alpha\beta)Z_n^{(b)}(kr) \; -$$

$$\frac{1}{r^2}\left[-n(n+1)\left(\frac{i\omega\epsilon + \sigma + i\beta\alpha/(\omega\mu)}{(i\omega\epsilon_r + \sigma_r) + (i\alpha_r\beta_r)/(\omega\mu_r)}\right)\right] Z_n^{(b)}(kr) = 0 \tag{1.6.48}$$

where k_a is defined by (1.6.11) and k is defined by (1.6.1). Letting ζ_b be defined by

$$\zeta_b = \left(\frac{i\omega\epsilon + \sigma + i\beta\alpha/(\omega\mu)}{(i\omega\epsilon_r + \sigma_r) + (i\alpha_r\beta_r)/(\omega\mu_r)}\right) \tag{1.6.49}$$

Substituting equation (1.6.49) into equation (1.6.31) we deduce that

$$c_{(m,n)}Z_n^{(c)} = -n(n+1)\zeta_b Z_n^{(b)} b_{(m,n)} \tag{1.6.50}$$

The equation (1.6.49) is substituted into equation (1.6.48) to yield the equation,

$$\left(\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\right)(rZ_n^{(b)}(kr))$$
$$+\left[(k_a^2+\alpha\beta) - \frac{n(n+1)\zeta_b}{r^2}\right]Z_n^{(b)}(kr) = 0 \tag{1.6.51}$$

where k_a is defined by (1.6.11) and k is defined by (1.6.1)

Combinations of solutions of equations (1.6.39) and (1.6.51) and their derivatives are used to represent the electric and magnetic fields induced inside an N layered sphere where each layer has nontrivial magnetic properties and the electric and magnetic properties are coupled in the sense that the layers are bianisotropic.

2 Expansion Coefficient Relations

2.1 Representations of E and H

Substituting equation (1.6.31) into (1.6.2) and making use of the relation defined by equation (1.6.49) and the modified propagation constant k defined by (1.6.1) we see that we can satisfy the Faraday and Ampere Maxwell equations for the special class of bianisotropic spheres treated in the previous section with an electric vector of the form,

$$\vec{E} = \sum_{(m,n)\in\mathcal{I}} \left\{ a_{(m,n)}Z_n^{(a)}(kr)\vec{A}_{(m,n)}(\theta,\phi) \quad + \right.$$
$$[-n(n+1)\{\zeta_b\}]b_{(m,n)}\frac{Z_n^{(b)}(kr)}{kr}\vec{C}_{(m,n)}(\theta,\phi) +$$
$$\left. \frac{b_{(m,n)}}{kr}\left(-\left(\frac{\partial}{\partial r}\right)(rZ_n^{(b)}(kr))\right)\vec{B}_{(m,n)}(\theta,\phi)\right\} \tag{2.1.1}$$

where the radial functions $Z_n^{(a)}$ and $Z_n^{(b)}$ satisfy equations (1.6.39) and (1.6.51).

Now making use of a form of the relation (1.6.21) given by

$$\left(\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\right)(rZ_n^{(b)}(kr)) =$$

$$+\left[\frac{n(n+1)\zeta_b}{r^2} - (k_a^2 + \alpha\beta)\right] Z_n^{(b)}(kr), \tag{2.1.2}$$

where k_a is defined by (1.6.11) and k is defined by (1.6.1) and its square is equal to the square of k_a plus $\alpha\beta$, we will be able to simplify the equation,

$$curl(\vec{E}) = \sum_{(m,n)\in\mathcal{I}} \Bigg\{$$

$$a_{(m,n)}\left[n(n+1)\frac{Z_n^{(a)}(kr)}{r}\vec{C}_{(m,n)} + \frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right)\vec{B}_{(m,n)}\right] +$$

$$(-n(n+1)\zeta_b)b_{(m,n)}\frac{Z_n^{(b)}(kr)}{kr^2}\vec{A}_{(m,n)} +$$

$$b_{(m,n)}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2(rZ_n^{(b)}(kr))\vec{A}_{(m,n)}\Bigg\} \tag{2.1.3}$$

In fact, substituting equation (2.1.2) into equation (2.1.3) we see that

$$curl(\vec{E}) =$$

$$\sum_{(m,n)\in\mathcal{I}}\Bigg\{a_{(m,n)}\left[n(n+1)\frac{Z_n^{(a)}(r)}{r}\vec{C}_{(m,n)} + \frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right)\vec{B}_{(m,n)}\right] +$$

$$(-n(n+1)\zeta_b)b_{(m,n)}\frac{Z_n^{(b)}(kr)}{kr^2}\vec{A}_{(m,n)}+$$

$$b_{(m,n)}\frac{1}{kr}\left[\frac{n(n+1)\zeta_b}{r^2} - (k_a^2 + \alpha\beta)\right] Z_n^{(b)}(kr)\vec{A}_{(m,n)}\Bigg\} \tag{2.1.4}$$

where k_a is given by (1.6.11) and k by equation (1.6.1).

Some telescoping in the right side of equation (2.1.4) yields the reduced form,

$$curl(\vec{E}) =$$

$$\sum_{(m,n)\in\mathcal{I}}\Bigg\{a_{(m,n)}\left[n(n+1)\frac{Z_n^{(a)}(kr)}{r}\vec{C}_{(m,n)} + \frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right)\vec{B}_{(m,n)}\right] +$$

$$-b_{(m,n)}\frac{1}{r}\left[(k_a^2 + \alpha\beta)\right] Z_n^{(b)}(kr)\vec{A}_{(m,n)}\Bigg\}$$

$$= i\omega\overline{\overline{\mu}}\vec{H} - \overline{\overline{\beta}}\vec{E} \tag{2.1.5}$$

again with k and k_a defined by (1.6.1) and (1.6.11), respectively.

Defining a new function $W_n^{(a)}$ by the rule

$$W_n^{(a)}(kr) = \frac{1}{kr}\left(\frac{\partial}{\partial r}\right)(rZ_n^{(a)}(kr)) \tag{2.1.6}$$

or equivalently by

$$W_n^{(a)}(kr) = \underset{z \to kr}{LIM} \left(\frac{1}{z}\right)\frac{\partial}{\partial z}(z\Psi_\nu^{(a)}(z)) \tag{2.1.7}$$

where $\Psi_\nu^{(a)}$ is defined by (1.6.44) and where ζ_a is related to the parameter ν in equation (2.1.7) by equation (1.6.38). We define $W_n^{(b)}(r)$ by changing a to b in equation (2.1.7).

Using the new function $W_n^{(a)}$ defined by equation (2.1.7) we define

$$curl(\vec{E}) =$$

$$\sum_{(m,n)\in\mathcal{I}}\left\{a_{(m,n)}\left[n(n+1)\frac{Z_n^{(a)}(kr)}{r}\vec{C}_{(m,n)} + a_{(m,n)}kW_n^{(a)}(kr)\vec{B}_{(m,n)}\right] + \right.$$

$$\left. -b_{(m,n)}\frac{1}{r}\left[(k_a^2+\alpha\beta)\right]Z_n^{(b)}(kr)\vec{A}_{(m,n)}\right\}$$

$$= i\omega\overline{\overline{\mu}}\vec{H} - \overline{\overline{\beta}}\vec{E} \tag{2.1.8}$$

where k and k_a are defined by (1.6.1) and (1.6.11), respectively.

In terms of the function $W_n^{(a)}(kr)$ we express the function $\vec{H}$ by the rule,

$$i\omega\overline{\overline{\mu}}\vec{H} =$$

$$\sum_{(m,n)\in\mathcal{I}}\left\{a_{(m,n)}\left[n(n+1)\frac{Z_n^{(a)}(kr)}{r}\vec{C}_{(m,n)} + a_{(m,n)}kW_n^{(a)}(kr)\vec{B}_{(m,n)}\right] + \right.$$

$$\left. -b_{(m,n)}\frac{1}{r}\left[(k_a^2+\alpha\beta)\right]Z_n^{(b)}(kr)\vec{A}_{(m,n)}\right\} +$$

$$\sum_{(m,n)\in\mathcal{I}}\left\{\quad a_{(m,n)}\beta Z_n^{(a)}(kr)\vec{A}_{(m,n)}(\theta,\phi) + \right.$$

$$\beta_r[-n(n+1)\{\zeta_b\}]b_{(m,n)}\frac{Z_n^{(b)}(kr)}{kr}\vec{C}_{(m,n)}(\theta,\phi) +$$

$$\left.\beta\frac{b_{(m,n)}}{kr}\left(-\left(\frac{\partial}{\partial r}\right)(rZ_n^{(b)}(kr))\right)\vec{B}_{(m,n)}(\theta,\phi)\right\} \tag{2.1.9}$$

with k being given by (1.6.1) and k_a by (1.6.11)

Collecting terms we find that equation (1.6.31) which relates the function $c_{(m,n)}Z_n^{(c)}$ to the function $Z_n^{(b)}$ can be used to derive the relationship,

$$
\begin{aligned}
&-i\omega\overline{\overline{\mu}}\vec{H} = \\
&\sum_{(m,n)\in\mathcal{I}} \left\{ \left[a_{(m,n)} n(n+1) \frac{Z_n^{(a)}(kr)}{r} + \alpha_r c_{(m,n)} \frac{Z_n^{(c)}(kr)}{kr} \right] \vec{C}_{(m,n)} \right. \\
&+ \left[a_{(m,n)} \frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(kr)\right) - \alpha b_{(m,n)} \left(\frac{1}{kr}\right)\left(\frac{\partial}{\partial r}\right)\left(rZ_n^{(b)}(kr)\right) \right] \vec{B}_{(m,n)} + \\
&\left. \left[c_{(m,n)} \frac{Z_n^{(c)}(kr)}{kr^2} + b_{(m,n)} \frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2 \left(rZ_n^{(b)}(kr)\right) + \alpha a_{(m,n)} Z_n^{(a)}(kr) \right] \vec{A}_{(m,n)} \right\}
\end{aligned}
\tag{2.1.10}
$$

and we could then use the differential equation (2.1.2) to simplify equation (2.1.10).

So far we have been trying to develop representations of the electric and magnetic vector in a special class of bianisotropic spheres. Let us now consider an N layered sphere and let k_p denote the propagation constant in the pth layer given by

$$k_p^2 = \omega^2 \mu^{(p)} \epsilon^{(p)} - i\omega\mu^{(p)}\sigma^{(p)} + \alpha^{(p)}\beta^{(p)} \tag{2.1.11}$$

where for the layer with index p, where p runs from 1 to N for the actual layers of the sphere and where $N+1$ is the region outside the sphere, and where $\mu^{(p)}$, $\epsilon^{(p)}$, $\sigma^{(p)}$, $\alpha^{(p)}$ and $\beta^{(p)}$ are respectively the tangential components of (i) the magnetic permeability, (ii) the permittivity, (iii) the conductivity, (iv) the Faraday Maxwell equation coupling tensor, and (v) the Maxwell equation coupling tensor, where these five tensors all have the same form as that given in equation (1.6.4).

Let us develop the full theory using the functions,

$$W_{(n,p)}^{(a,j)}(k_p r) = \frac{1}{k_p r}\left(\frac{\partial}{\partial r}\right)\left(rZ_{(n,p)}^{(a,j)}(k_p r)\right) \tag{2.1.12}$$

where the propagation constant k_p is given by (2.1.11) and where $Z_{(n,p)}^{(a,j)}(k_p r)$ is the singular solution if $j=3$ and the solution with the integrable singularity at $r=0$ corresponds to $j=1$.

The expansion coefficients in layer p associated with the functions, $Z_{(n,p)}^{(a,1)}(k_p r)$ and with the functions $W_{(n,p)}^{(a,1)}(k_p r)$ which have the integrable singularity at the origin will be

denoted by $a^{(p)}_{(m,n)}$ and $b^{(p)}_{(m,n)}$ and the coefficients and $\alpha^{(p)}_{(m,n)}$ and $\beta^{(p)}_{(m,n)}$ will be multipliers of the functions $Z^{(a,3)}_{(n,p)}(k_p r)$ and $W^{(a,3)}_{(n,p)}(k_p r)$ which are singular at $r = 0$. The electric vector with general representation given by equation (1.6.2) is in the pth layer of the multilayer bianisotropic sphere represented by

$$\vec{E}_p =$$

$$= \sum_{(m,n)\in\mathcal{I}} \Bigg\{ \quad [a^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p)}(k_p r) + \alpha^{(p)}_{(m,n)} Z^{(a,3)}_{(n,p)}(k_p r)]\vec{A}_{(m,n)}(\theta,\phi) +$$

$$[-n(n+1)\,\{\zeta_b\}]b^{(p)}_{(m,n)} \frac{Z^{(a,1)}_{(n,p)}(k_p r)}{kr}]\vec{C}_{(m,n)}(\theta,\phi) +$$

$$[-n(n+1)\,\{\zeta_b\}]\beta^{(p)}_{(m,n)} \frac{Z^{(a,3)}_{(n,p)}(k_p r)}{k_p r}]\vec{C}_{(m,n)}(\theta,\phi) +$$

$$[-b^{(p)}_{(m,n)} W^{(b,1)}_{(n,p)}(k_p r) - \beta^{(p)}_{(m,n)} W^{(b,3)}_{(n,p)}(k_p r)]\vec{B}_{(m,n)}(\theta,\phi) \quad \Bigg\} \tag{2.1.13}$$

where k_p is defined by (2.1.11).

Using our previous expression for the magnetic field vector but using the definitions (2.1.12) and the fact that the k_p defined by (2.1.11) is the propagation constant in the pth layer, we see that the Ampere Maxwell equation with a coupling tensor defined by (1.6.4) the magnetic vector in the innermost layer with p equal to 1 has the form,

$$\vec{H} = \sum_{(m,n)\in\mathcal{I}} \Bigg[$$

$$\frac{i}{\omega\mu_r}\left\{a^{(p)}_{(m,n)} \frac{Z^{(a,1)}_{(n,p)}(k\ r)n(n+1)}{r} + \alpha_r c^{(p)}_{(m,n)} \frac{Z^{(c,1)}_{(n,p)}(k_p r)}{k_p r}\right\}\vec{C}_{(m,n)}$$

$$+\frac{i}{\omega\mu}\left\{a^{(p)}_{(m,n)} k W^{(a,1)}_{(n,p)}(k_p r) + \alpha b^{(p)}_{(m,n)} \frac{1}{kr}\left(-\frac{\partial}{\partial r}\right)\left(r Z^{(b,1)}_{(n,p)}(k_p r)\right)\right\}\vec{B}_{(m,n)} +$$

$$\left(\frac{i}{\omega\mu}\right)\left\{c^{(p)}_{(m,n)} \frac{Z^{(c,1)}_{(n,p)}(k_p r)}{k_p r^2} +\right.$$

$$\left. b^{(p)}_{(m,n)}\left(\frac{1}{kr}\right)\left(\frac{\partial}{\partial r}\right)^2 \left(r Z^{(b,1)}_{(n,p)}(k_p r)\right) + \alpha a^{(p)}_{(m,n)} Z^{((a,1))}_{(n,p)}(k_p r)\right\}\vec{A}_{(m,n)}\Bigg] \tag{2.1.14}$$

where k_p is given by (2.1.11) Now using equation (1.6.50) and equation (1.6.20) we see that equation (2.1.14) can be simplified by the telescoping of terms and specifically making use of the relation that is derivable from equations (1.6.50) and (1.6.51) given by

$$c^{(p)}_{(m,n)} \frac{Z^{(c,1)}_{(n,p)}(k_p r)}{k_p r^2} +$$

$$b^{(p)}_{(m,n)} \left(\frac{1}{k_p r}\right) \left(\frac{\partial}{\partial r}\right)^2 \left(r Z^{(b,1)}_{(n,p)}(k_p r)\right) =$$

$$b^{(p)}_{(m,n)} \left[-\frac{(\omega^2 \mu^{(p)} \epsilon^{(p)} - i\omega\mu^{(p)}\sigma^{(p)} + \alpha^{(p)}\beta^{(p)})}{k_p} \right] Z^{(b,1)}_{(n,p)}(k_p r) \tag{2.1.15}$$

In doing this we see that the magnetic vector in the core of the multilayer spherical structure corresponding to $p = 1$ is given by

$$\vec{H} =$$

$$\sum_{(m,n)\in\mathcal{I}} \left[\frac{i}{\omega\mu_r^{(p)}} \left\{ a^{(p)}_{(m,n)} \frac{Z^{(a,1)}_{(n,p)}(k_p r) n(n+1)}{r} - \alpha_r \zeta_b n(n+1) b^{(p)}_{(m,n)} \frac{Z^{(b,1)}_{(n,p)}(k_p r)}{k_p r} \right\} \vec{C}_{(m,n)} \right.$$

$$+ \frac{i}{\omega\mu^{(p)}} \left(\left\{ a^{(p)}_{(m,n)} k_p W^{(a,1)}_{(n,p)}(k_p r) + \alpha b^{(p)}_{(m,n)} (-W^{(b,1)}_{(n,p)}(k_p r)) \right\} \vec{B}_{(m,n)} + \right.$$

$$\left. \left. \left\{ b^{(p)}_{(m,n)} \left[-k_p \right] Z^{(b,1)}_{(n,p)}(k_p r) + \alpha^{(p)} a^{(p)}_{(m,n)} Z^{((a,1))}_{(n,p)}(k_p r) \right\} \vec{A}_{(m,n)} \right) \right] \tag{2.1.16}$$

where k_p is defined by (2.1.11) and where we have made use of the relation,

$$k_p = \frac{\omega^2 \mu^{(p)} \epsilon^{(p)} - i\omega\mu^{(p)}\sigma^{(p)} + \alpha^{(p)}\beta^{(p)}}{k_p} \tag{2.1.17}$$

We now consider the representation of the magnetic vector in an interior layer of a multilayer sphere that does not contain the center of the sphere. The magnetic vector has the representation in terms of functions $Z^{(a,1)}_{(n,p)}$ and $Z^{(b,1)}_{(n,p)}$ which have integrable singularites at the origin, and the functions $Z^{(a,3)}_{(n,p)}$ and $Z^{(b,3)}_{(n,p)}$ whose representation, in the case considered here involves Hankel functions with complex index. The magnetic vector representation in a penetrable shell is given by

$$\vec{H} = \sum_{(m,n)\in\mathcal{I}} \left[\left(\frac{i}{\omega\mu_r^{(p)}}\right) \left\{ a^{(p)}_{(m,n)} \frac{Z^{(a,1)}_{(n,p)}(k_p r) n(n+1)}{r} + \right. \right.$$

$$\alpha^{(p)}_{(m,n)}\frac{Z^{(a,3)}_{(n,p)}(k_pr)n(n+1)}{r} + (-1)\left(\alpha^{(p)}_r\zeta_b n(n+1)b^{(p)}_{(m,n)}\frac{Z^{(b,1)}_{(n,p)}(k_pr)}{k_pr} + \right.$$

$$\left.\alpha^{(p)}_r\zeta_b n(n+1)\beta^{(p)}_{(m,n)}\frac{Z^{(b,3)}_{(n,p)}(k_pr)}{k_pr}\right)\Bigg\}\vec{C}_{(m,n)} +$$

$$\frac{i}{\omega\mu^{(p)}}\left\{a^{(p)}_{(m,n)}k_pW^{(a,1)}_{(n,p)}(k_pr) + \alpha^{(p)}_{(m,n)}k_pW^{(a,3)}_{(n,p)}(k_pr)\right\}\vec{B}_{(m,n)} +$$

$$\left(\frac{-i}{\omega\mu^{(p)}}\right)\left\{\alpha^{(p)}b^{(p)}_{(m,n)}(W^{(b,1)}_{(n,p)}(k_pr)) + \alpha^{(p)}\beta^{(p)}_{(m,n)}(W^{(b,3)}_{(n,p)}(k_pr))\right\}\vec{B}_{(m,n)} +$$

$$\left(\frac{i}{\omega\mu^{(p)}}\right)[-k_p]\left\{b^{(p)}_{(m,n)}Z^{(b,1)}_{(n,p)}(k_pr) + \beta^{(p)}_{(m,n)}Z^{(b,3)}_{(n,p)}(k_pr)\right\}\vec{A}_{(m,n)} +$$

$$\left(\frac{i}{\omega\mu^{(p)}}\right)\left\{\alpha^{(p)}a^{(p)}_{(m,n)}Z^{(a,1)}_{(n,p)}(k_pr) + \alpha^{(p)}\alpha^{(p)}_{(m,n)}Z^{(a,3)}_{(n,p)}(k_pr)\right\}\vec{A}_{(m,n)}\Bigg] \tag{2.1.18}$$

where k_p is given by equation (2.1.11) and we have made use of equation (2.1.17).

We now consider the representation of the electric vector in the core region $p = 1$ of the multilayer, spherically symmetric bianisotropic structure. Making use of equation (1.6.51) we deduce from equation (1.6.2) that

$$\vec{E}_p = \sum_{(m,n)\in\mathcal{I}}\Bigg\{a_{(m,n)}Z^{(a,1)}_{(n,p)}(k_pr)\vec{A}_{(m,n)}(\theta,\phi) \quad +$$

$$[-n(n+1)\{\zeta_b\}]b_{(m,n)}\frac{Z^{(b,1)}_{(n,p)}(k_pr)}{k_pr}\vec{C}_{(m,n)}(\theta,\phi) +$$

$$\frac{b_{(m,n)}}{k_pr}\left(-\left(\frac{\partial}{\partial r}\right)(rZ^{(b,1)}_{(n,p)}(k_pr))\right)\vec{B}_{(m,n)}(\theta,\phi)\Bigg\} \tag{2.1.19}$$

where k_p is given by (2.1.11).

Equating tangential components of $\vec{E}$ across the shell $r = R_p$ equation (2.1.13), the representation of the electric vector in a shell region, implies that equating coefficients of $\vec{A}_{(m,n)}(\theta,\phi)$ leads, for r equal to R_p to the relation,

$$[a^{(p)}_{(m,n)}Z^{(a,1)}_{(n,p)}(k_pr) + \alpha^{(p)}_{(m,n)}Z^{(a,3)}_{(n,p)}(k_pr)]$$

$$= [a^{(p+1)}_{(m,n)}Z^{(a,1)}_{(n,p+1)}(k_{p+1}r) + \alpha^{(p+1)}_{(m,n)}Z^{(a,3)}_{(n,p+1)}(k_{p+1}r)] \tag{2.1.20}$$

Multiplying both sides of equation (2.1.13) by $\vec{B}_{(m,n)}$ and integrating over the sphere $r = R_p$, we deduce, for r equal to R_p that

$$[b^{(p)}_{(m,n)}Z^{(b,1)}_{(n,p)}(k_pr) + \beta^{(p)}_{(m,n)}Z^{(b,3)}_{(n,p)}(k_pr)]$$

$$= [b^{(p+1)}_{(m,n)} Z^{(b,1)}_{(n,p+1)}(k_{p+1}r) + \beta^{(p+1)}_{(m,n)} Z^{(b,3)}_{(n,p+1)}(k_{p+1}r)] \tag{2.1.21}$$

We now set up the differential equations which state that the tangential components of the magnetic vector are continuous across the boundary of a sphere separating regions of continuity of tensorial electric properties. Equation (2.1.18) implies, upon equating tangential components $\vec{H}$, on each side of the boundary $r = R_p$, by taking the dot product of both sides of (2.1.18) with respect to $\vec{B}_{(m,n)}$ and integrating over the sphere $r = R_p$ that

$$\frac{i}{\omega\mu^{(p)}} \left\{ a^{(p)}_{(m,n)} k_p W^{(a,1)}_{(n,p)}(k_p r) + \alpha^{(p)}_{(m,n)} k_p W^{(a,3)}_{(n,p)}(k_p r) \right\} +$$

$$\left(\frac{-i}{\omega\mu^{(p)}}\right) \left\{ \alpha^{(p)} b^{(p)}_{(m,n)} (W^{(b,1)}_{(n,p)}(k_p r)) \ + \ \alpha^{(p)} \beta^{(p)}_{(m,n)} (W^{(b,3)}_{(n,p)}(k_p r)) \right\} =$$

$$\frac{i}{\omega\mu^{(p+1)}} \left\{ a^{(p+1)}_{(m,n)} k_{p+1} W^{(a,1)}_{(n,p+1)}(k_{p+1} r) + \alpha^{(p+1)}_{(m,n)} k_{p+1} W^{(a,3)}_{(n,p+1)}(k_{p+1} r) \right\} +$$

$$\left(\frac{-i}{\omega\mu^{(p+1)}}\right) \left\{ \alpha^{(p+1)} b^{(p+1)}_{(m,n)} (W^{(b,1)}_{(n,p+1)}(k_{p+1} r)) + \alpha^{(p+1)} \beta^{(p+1)}_{(m,n)} (W^{(b,3)}_{(n,p+1)}(k_{p+1} r)) \right\} \tag{2.1.22}$$

Using equation (2.1.18) and equating coefficients of the vector $\vec{A}$ on both sides of the sphereical shell $r = R_p$ we have

$$\left(\frac{i}{\omega\mu^{(p)}}\right) [-k_p] \left\{ b^{(p)}_{(m,n)} Z^{(b,1)}_{(n,p)}(k_p r) + \beta^{(p)}_{(m,n)} Z^{(b,3)}_{(n,p)}(k_p r) \right\}$$

$$\left(\frac{i}{\omega\mu^{(p)}}\right) \left\{ \alpha^{(p)} a^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p)}(k_{p+1} r) + \alpha^{(p)} \alpha^{(p)}_{(m,n)} Z^{(a,3)}_{(n,p)}(k_{p+1} r) \right\} =$$

$$\left(\frac{i}{\omega\mu^{(p+1)}}\right) \left\{ b^{(p+1)}_{(m,n)} [-k_{p+1}] \, Z^{(b,1)}_{(n,-+1)}(k_{p+1} r) \right.$$

$$\left. + \ \beta^{(p+1)}_{(m,n)} [--k_{p+1}] \, Z^{(b,3)}_{(n,p+1)}(k_{p+1} r) \right\}$$

$$\left(\frac{i}{\omega\mu^{(p+1)}}\right) \left\{ \alpha^{(p+1)} a^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p+1)}(k_{p+1} r) + \alpha^{(p+1)} \alpha^{(p+1)}_{(m,n)} Z^{(a,3)}_{(n,p+1)}(k_{p+1} r) \right\} \tag{2.1.23}$$

2.2 Transition Matrices

We now attempt to develop transition matrices which will relate expansion coefficients in one layer to expansion coefficients in another layer. We start with equation (2.1.22);

we find, after multiplying both sides of this equation by $\mu^{(p)}$ and dividing both sides of equation (2.1.22) by k_p, that

$$\left\{a^{(p)}_{(m,n)} W^{(a,1)}_{(n,p)}(k_p r) + \alpha^{(p)}_{(m,n)} W^{(a,3)}_{(n,p)}(k_p r)\right\} +$$

$$\left\{(-\frac{\alpha^{(p)}}{k_p}) b^{(p)}_{(m,n)} (W^{(b,1)}_{(n,p)}(k_p r)) \ + \ (-\frac{\alpha^{(p)}}{k_p}) \beta^{(p)}_{(m,n)} (W^{(b,3)}_{(n,p)}(k_p r))\right\} =$$

$$\left(\frac{\mu^{(p)} k_{p+1}}{\mu^{(p+1)} k_p}\right) \left\{a^{(p+1)}_{(m,n)} W^{(a,1)}_{(n,p+1)}(k_p r) + \alpha^{(p+1)}_{(m,n)} W^{(a,3)}_{(n,p+1)}(k_p r)\right\} +$$

$$\left(\frac{-\mu^{(p)} \alpha^{(p+1)}}{\mu^{(p+1)} k_p}\right) \left\{b^{(p+1)}_{(m,n)} (W^{(b,1)}_{(n,p+1)}(k_p r)) \ + \ \beta^{(p+1)}_{(m,n)} (W^{(b,3)}_{(n,p+1)}(k_p r))\right\} \tag{2.2.1}$$

Multiplying both sides of equation (2.1.18) by $\vec{A}_{(m,n)}(\theta, \phi)$ and observing that

$$\underset{r \to R_p^-}{LIM} \int_{S_p(r)} \vec{H} \cdot \vec{A}_{(m,n)}(\theta, \phi) dA =$$

$$\underset{r \to R_p^+}{LIM} \int_{S_p(r)} \vec{H} \cdot \vec{A}_{(m,n)}(\theta, \phi) dA \tag{2.2.2}$$

we derive equation (2.1.23). From this, after multiplying all terms by $-i\omega\mu^{(p)} k_p$ and dividing all terms by k_p^2, where k_p is defined by (2.1.11), we derive the relation that

$$\left(\frac{\alpha^{(p)}}{k_p}\right) \left\{a^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p)}(k_p r) + \alpha^{(p)}_{(m,n)} Z^{(a,3)}_{(n,p)}(k_p r)\right\} \ +$$

$$\left\{b^{(p)}_{(m,n)} Z^{(b,1)}_{(n,p)}(k_p r) + \beta^{(p)}_{(m,n)} Z^{(b,3)}_{(n,p)}(k_p r)\right\} \ =$$

$$\left(\frac{\mu^{(p)} \alpha^{(p+1)}}{\mu^{(p+1)} k_p}\right) \left\{a^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p+1)}(k_{p+1} r) + \alpha^{(p+1)}_{(m,n)} Z^{(a,3)}_{(n,p+1)}(k_{p+1} r)\right\} \ +$$

$$\left[-\frac{\mu^{(p)} k_{p+1}}{\mu^{(p+1)} k_p}\right] \cdot \left\{b^{(p+1)}_{(m,n)} Z^{(b,1)}_{(n,p+1)}(k_{p+1} r) + \beta^{(p+1)}_{(m,n)} Z^{(b,3)}_{(n,p+1)}(k_{p+1} r)\right\}. \tag{2.2.3}$$

where k_p and k_{p+1} are defined by (2.1.11).

We now define parameters which appear in the matrix relating expansion coefficients in one layer to those in an adjacent layer. We obtain these by considering terms appearing in equation (2.2.3)

$$\rho^{(p+1)}_{(b,3)} \ = \ \left(\frac{\mu^{(p)}}{\mu^{(p+1)}}\right) \left[-\frac{(k_{p+1}}{k_p}\right] \tag{2.2.4}$$

Also

$$\rho^{(p+1)}_{(a,3)} = \left(\frac{\mu^{(p)}\alpha^{(p+1)}}{\mu^{(p+1)}k_p}\right) \tag{2.2.5}$$

with k_p and k_{p+1} being defined by (2.1.11). A similar term appearing in the inner shell matrix is

$$\rho^{(p)}_{(a,3)} = \left(\frac{\alpha^{(p)}}{k_p}\right) \tag{2.2.6}$$

A term in the second row of the outer shell matrix is

$$\rho^{(p+1)}_{(a,2)} = \left(\frac{\mu^{(p)}k_{p+1}}{\mu^{(p+1)}k_p}\right) \tag{2.2.7}$$

Another term appearing in second row of the matrix is

$$\rho^{(p+1)}_{(b,2)} = \left(\frac{-\mu^{(p)}\alpha^{(p+1)}}{\mu^{(p+1)}k_p}\right) \tag{2.2.8}$$

The corresponding term in the inner shell matrix is

$$\rho^{(p)}_{(b,2)} = \left(\frac{-\alpha^{(p)}}{k_p}\right) \tag{2.2.9}$$

With the special functions $Z^{(a,j)}_{(n,p)}$, defined by (1.6.39), and $Z^{(b,j)}_{(n,p)}$, defined by (1.6.51), and the derivative terms defined by equation (2.1.12) being evaluated at the separating spherical boundary $r = R_p$, we see that the matrix equation relating expansion coefficients in layer p to those in layer $p+1$ is given by

$$\begin{bmatrix} Z^{(a,1)}_{(n,p)} & Z^{(a,3)}_{(n,p)} & 0 & 0 \\ W^{(a,1)}_{(n,p)} & W^{(a,3)}_{(n,p)} & \rho^{(p)}_{(b,2)}W^{(b,1)}_{(n,p)} & \rho^{(p)}_{(b,2)}W^{(b,3)}_{(n,p)} \\ \rho^{(p)}_{(a,3)}Z^{(a,1)}_{(n,p)} & \rho^{(p)}_{(a,3)}Z^{(a,3)}_{(n,p)} & Z^{(b,1)}_{(n,p)} & Z^{(b,3)}_{(n,p)} \\ 0 & 0 & W^{(b,1)}_{(n,p)} & W^{(b,3)}_{(n,p)} \end{bmatrix} \begin{bmatrix} a^{(p)}_{(m,n)} \\ \alpha^{(p)}_{(m,n)} \\ b^{(p)}_{(m,n)} \\ \beta^{(p)}_{(m,n)} \end{bmatrix} =$$

$$\begin{bmatrix} Z^{(a,1)}_{(n,p+1)} & Z^{(a,3)}_{(n,p+1)} & 0 & 0 \\ W^{(a,1)}_{(n,p+1)} & W^{(a,3)}_{(n,p+1)} & \rho^{(p+1)}_{(b,2)}W^{(b,1)}_{(n,p+1)} & \rho^{(p+1)}_{(b,2)}W^{(b,3)}_{(n,p+1)} \\ \rho^{(p+1)}_{(a,3)}Z^{(a,1)}_{(n,p+1)} & \rho^{(p+1)}_{(a,3)}Z^{(a,3)}_{(n,p+1)} & \rho^{(p+1)}_{(b,3)}Z^{(b,1)}_{(n,p+1)} & \rho^{(p+1)}_{(b,3)}Z^{(b,3)}_{(n,p+1)} \\ 0 & 0 & W^{(b,1)}_{(n,p+1)} & W^{(b,3)}_{(n,p+1)} \end{bmatrix} \begin{bmatrix} a^{(p+1)}_{(m,n)} \\ \alpha^{(p+1)}_{(m,n)} \\ b^{(p+1)}_{(m,n)} \\ \beta^{(p+1)}_{(m,n)} \end{bmatrix} \tag{2.2.10}$$

This equation can be written more compactly in the form

$$T^{(p)}_n(k_pR_p)\begin{bmatrix} a^{(p)}_{(m,n)} \\ \alpha^{(p)}_{(m,n)} \\ b^{(p)}_{(m,n)} \\ \beta^{(p)}_{(m,n)} \end{bmatrix} = T^{(p+1)}_n(k_{p+1}R_p)\begin{bmatrix} a^{(p+1)}_{(m,n)} \\ \alpha^{(p+1)}_{(m,n)} \\ b^{(p+1)}_{(m,n)} \\ \beta^{(p+1)}_{(m,n)} \end{bmatrix} \tag{2.2.11}$$

To compute the inverse of the matrix $T_n^{(p)}$ we need its transpose which is given by

$$\begin{bmatrix} Z_{(n,p)}^{(a,1)}(k_pR_p) & W_{(n,p)}^{(a,1)}(k_pR_p) & \rho_{(a,3)}^{(p)}Z_{(n,p)}^{(a,1)}(k_pR_p) & 0 \\ Z_{(n,p)}^{(a,3)}(k_pR_p) & W_{(n,p)}^{(a,3)}(k_pR_p) & \rho_{(a,3)}^{(p)}Z_{(n,p)}^{(a,3)}(k_pR_p) & 0 \\ 0 & \rho_{(b,2)}^{(p)}W_{(n,p)}^{(b,1)}(k_pR_p) & Z_{(n,p)}^{(b,1)}(k_pR_p) & W_{(n,p)}^{(b,1)}(k_pR_p) \\ 0 & \rho_{(b,2)}^{(p)}W_{(n,p)}^{(b,3)}(k_pR_p) & Z_{(n,p)}^{(b,3)}(k_pR_p) & W_{(n,p)}^{(b,3)}(k_pR_p) \end{bmatrix} = (T_n^{(p)})^{transpose})^t \tag{2.2.12}$$

Wronskian relations will show that we can define a new matrix $Q_n^{(p)}$ by the rule

$$Q_n^{(p)} = T_n^{(p)}(k_pR_p)^{-1}T_n^{(p+1)}(k_{p+1}R_p). \tag{2.2.13}$$

Using equations (2.2.11) and (2.2.13) we see that the expansion coefficients in the core are related to the expansion coefficients in the outer shell by the rule,

$$\begin{bmatrix} a_{(m,n)}^{(1)} \\ 0 \\ b_{(m,n)}^{(1)} \\ 0 \end{bmatrix} = Q_n^{(1)}Q_n^{(2)}\cdots Q_n^{(N)} \begin{bmatrix} a_{(m,n)}^{(N+1)} \\ \alpha_{(m,n)}^{(N+1)} \\ b_{(m,n)}^{(N+1)} \\ \beta_{(m,n)}^{(N+1)} \end{bmatrix} \tag{2.2.14}$$

This gives us four equations in four unknowns, since we assume that the expansion coefficients $\alpha_{(m,n)}^{(N+1)}$ and $\beta_{(m,n)}^{(N+1)}$ are determined; these expansion coefficients could define a complex source such as a radar or laser beam in the near field (Barton [8] and [9], Pinnick [39] and [37]). Solving equation (2.2.14) we find values of $a_{(m,n)}^{(1)}$ and $b_{(m,n)}^{(1)}$ and assuming that $\alpha_{(m,n)}^{(1)}$ and $\beta_{(m,n)}^{(1)}$ are both zero, we can easily obtain the expansion coefficients in every layer of the structure. If we define the matrix $\mathcal{R}_n^{(p)}$ by the rule,

$$\mathcal{R}_n^{(p)} = T_n^{(p+1)}(k_{p+1}R_p)^{-1}T_n^{(p)}(k_pR_p) \tag{2.2.15}$$

We see that the definition of $\mathcal{R}_n^{(p)}$ by equation (2.2.15) implies the relationship

$$\mathcal{R}_n^{(p)} \begin{bmatrix} a_{(m,n)}^{(p)} \\ \alpha_{(m,n)}^{(p)} \\ b_{(m,n)}^{(p)} \\ \beta_{(m,n)}^{(p)} \end{bmatrix} = \begin{bmatrix} a_{(m,n)}^{(p+1)} \\ \alpha_{(m,n)}^{(p+1)} \\ b_{(m,n)}^{(p+1)} \\ \beta_{(m,n)}^{(p+1)} \end{bmatrix} \tag{2.2.16}$$

between expansion coefficients in adjacent layers of the spherical structure.

These computations using equation (2.2.16) are facilitated by the fact that we have exact formulas for the determinant and inverses of the 4 by 4 matrices $T_n^{(p)}$ Let the determinant of $T_n^{(p)}$ be defined by

$$
\begin{aligned}
\Delta_p = {} & Z_{(n,p)}^{(a,1)}(k_pR_p)W_{(n,p)}^{(a,3)}(k_pR_p) \\
& \left\{Z_{(n,p)}^{(b,1)}(k_pR_p)W_{(n,p)}^{(b,3)}(k_pR_p) - W_{(n,p)}^{(b,1)}(k_pR_p)Z_{(n,p)}^{(b,3)}(k_pR_p)\right\} + \\
& (-1)\left[Z_{(n,p)}^{(a,3)}(k_pR_p)W_{(n,p)}^{(a,1)}(k_pR_p)\right] \\
& \left\{Z_{(n,p)}^{(b,1)}(k_pR_p)W_{(n,p)}^{(b,3)}(k_pR_p) - W_{(n,p)}^{(b,1)}(k_pR_p)Z_{(n,p)}^{(b,3)}(k_pR_p)\right\}
\end{aligned}
\tag{2.2.17}
$$

which means that the determinant Δ_p is the product of two Wronskians $\mathcal{W}_{(n,p)}^{(a)}$ and $\mathcal{W}_{(n,p)}^{(b)}$ where

$$
\mathcal{W}_{(n,p)}^{(b)} = Z_{(n,p)}^{(b,1)}(k_pR_p)W_{(n,p)}^{(b,3)}(k_pR_p) - W_{(n,p)}^{(b,1)}(k_pR_p)Z_{(n,p)}^{(b,3)}(k_pR_p) \tag{2.2.18}
$$

We find that equation (2.2.17) and the Wronskian relationship,

$$
\mathcal{W}_{(n,p)}^{(a)}(k_pR_p) = \frac{-i}{(k_pR_p)^2} \tag{2.2.19}
$$

enables us to compute determinants with no roundoff error. This enables us to get exact formulas for the entries of the inverse of this matrix. If $(T_n^{p)}(k_pR_p)^{-1})_{(i,j)}$ denotes the entry in the ith row and jth column of the inverse of the matrix $T_n^{(p)}$, then the entry in row 1 and column 1 of the inverse is

$$
(T_n^{p)}(k_pR_p)^{-1})_{(1,1)} = W_{(n,p)}^{(a,3)}(k_pR_p)\mathcal{W}_{(n,p)}^{(b)}(k_pR_p)/\Delta_p, \tag{2.2.20}
$$

The (1,2) entry is

$$
(T_n^{p)}(k_pR_p)^{-1})_{(1,2)} = -Z_{(n,p)}^{(a,3)}(k_pR_p)\mathcal{W}_{(n,p)}^{(b)}(k_pR_p)/\Delta_p, \tag{2.2.21}
$$

The (1,3) term is

$$
(T_n^{p)}(k_pR_p)^{-1})_{(1,3)} = 0 \tag{2.2.22}
$$

The (1,4) term is

$$
(T_n^{p)}(k_pR_p)^{-1})_{(1,4)} = -(Z_{(n,p)}^{(a,3)}(k_pR_p)\left(\frac{-\alpha^{(p)}}{k_p}\right)\mathcal{W}_{(n,p)}^{(b)}(k_pR_p))/\Delta_p, \tag{2.2.23}
$$

Equations (2.2.20), (2.2.21), (2.2.22), and (2.2.23) define the first row of the transition matrix. The entry in row 2 and column 1 of the inverse is

$$(T_n^{p)}(k_pR_p)^{-1})_{(2,1)} = -W_{(n,p)}^{(a,1)}(k_pR_p)\mathcal{W}_{(n,p)}^{(b)}(k_pR_p)/\Delta_p, \tag{2.2.24}$$

The entry in row 2 and column 2 of the inverse is

$$(T_n^{p)}(k_pR_p)^{-1})_{(2,2)} = Z_{(n,p)}^{(a,1)}(k_pR_p)\mathcal{W}_{(n,p)}^{(b)}(k_pR_p)/\Delta_p, \tag{2.2.25}$$

The entry in row 2 and column 3 of the inverse is

$$(T_n^{p)}(k_pR_p)^{-1})_{(2,3)} = 0 \tag{2.2.26}$$

The entry in row 2 and column 4 of the inverse is

$$(T_n^{p)}(k_pR_p)^{-1})_{(2,4)} = (Z_{(n,p)}^{(a,1)}(k_pR_p)\left(\frac{-\alpha^{(p)}}{k_p}\right)\mathcal{W}_{(n,p)}^{(b)}(k_pR_p))/\Delta_p, \tag{2.2.27}$$

Equations (2.2.24), (2.2.25), (2.2.26), and (2.2.27) define the second row of the transition matrix. The (3,1) entry is

$$(T_n^{p)}(k_pR_p)^{-1})_{(3,1)} \;=\; W_{(n,p)}^{(b,3)}(k_pR_p)\left(\frac{-\alpha^{(p)}}{k_p}\right)\mathcal{W}_{(n,p)}^{(b)}(k_pR_p)/\Delta_p, \tag{2.2.28}$$

The (3,2) entry is

$$(T_n^{p)}(k_pR_p)^{-1})_{(3,2)} = 0 \tag{2.2.29}$$

The (3,3) entry is

$$(T_n^{p)}(k_pR_p)^{-1})_{(3,3)} = W_{(n,p)}^{(b,3)}(k_pR_p)\mathcal{W}_{(n,p)}^{(a)}(k_pR_p)/\Delta_p, \tag{2.2.30}$$

The (3,4) entry is

$$(T_n^{p)}(k_pR_p)^{-1})_{(3,4)} = -Z_{(n,p)}^{(b,3)}(k_pR_p)\mathcal{W}_{(n,p)}^{(a)}(k_pR_p)/\Delta_p, \tag{2.2.31}$$

Equations (2.2.28), (2.2.29), (2.2.30), and (2.2.31) define the third row of the matrix. The (4,1) entry is given by

$$(T_n^{p)}(k_pR_p)^{-1})_{(4,1)} \;=\; -\,W_{(n,p)}^{(b,1)}(k_pR_p)\left(\frac{-\alpha^{(p)}}{k_p}\right)\mathcal{W}_{(n,p)}^{(a)}(k_pR_p)/\Delta_p, \tag{2.2.32}$$

The (4,2) entry is

$$(T_n^{p)}(k_pR_p)^{-1})_{(4,2)} = 0 \tag{2.2.33}$$

The (4,3) entry is

$$(T_n^{p)}(k_pR_p)^{-1})_{(4,3)} = -W_{(n,p)}^{(b,1)}(k_pR_p)\mathcal{W}_{(n,p)}^{(a)}(k_pR_p)/\Delta_p, \tag{2.2.34}$$

Finally, the (4,4) entry of the inverse of $T_n^{(p)}$ is

$$(T_n^{p)}(k_pR_p)^{-1})_{(4,4)} = Z_{(n,p)}^{(b,1)}(k_pR_p)\mathcal{W}_{(n,p)}^{(a)}(k_pR_p)/\Delta_p, \tag{2.2.35}$$

We have therefore obtained round-off error free expressions for the entries of the inverse of $T_n^{(p)}(k_pR_p)$. Thus, except for the expression relating the expansion coefficients in equation (2.2.14), all computations are carried out by exact formulas. The matrix inverse computation requires no subtractions or additions and consequently there is no round off error if the Bessel and Hankel functions of complex index and their derivatives can be computed precisely.

2.3 Determination of Expansion Coefficients

Let us suppose that we have an N layer sphere subject to plane wave radiation. By multiplying the inverse of $T_n^{(p)}$ evaluated at k_pR_p by the matrix $T_n^{(p+1)}$ evaluated at $k_{p+1}R_p$ we obtaining the matrix

$$\mathcal{T}_n^{(p)} = T_n^{(p)}(k_pR_p)^{-1}T_n^{(p+1)}(k_{p+1}R_p) \tag{2.3.1}$$

relating the expansion coefficients in layer p to those in layer $p+1$. We then multiply all of these matrices (2.3.1) obtaining a matrix

$$\mathcal{T} = \mathcal{T}_n^{(1)} \cdot \mathcal{T}_n^{(2)} \cdots \mathcal{T}_n^{(N)} \tag{2.3.2}$$

where N is the number of layers of the sphere which relates the expansion coefficients in the core to the expansion coefficients in the space surrounding the sphere. This gives four equations in four unknowns. But it is really simpler than that. Using the second and fourth rows of this matrix equation, we can relate the expansion coefficients of the

scattered radiation to the known expansion coefficients of the incoming radiation. We then have in the first and third rows of this equation a formula for the expansion coefficients in the inner core.

3 Optical and Absorption Efficiency

3.1 Definition of Terms

The optial efficiency of a general N layer sphere exposed to plane wave radiation is defined to be

$$O_e = \left(\frac{Q_s + Q_a}{\| \vec{S^i} \|}\right)\left(\frac{1}{\pi R_N^2}\right) \tag{3.1.1}$$

where

$$\vec{S^i} = \text{the incoming radiation's Poynting vector} \tag{3.1.2}$$

and where

$$Q_a = \text{the total absorbed power} \tag{3.1.3}$$

and

$$Q_s = \text{the total scattered power} \tag{3.1.4}$$

and

$$R_N = \text{the radius of the outer shell} \tag{3.1.5}$$

The absorption efficiency is

$$A_e = \left(\frac{Q_a}{\| \vec{S^i} \|}\right)\left(\frac{1}{\pi R_N^2}\right) \tag{3.1.6}$$

These efficiencies O_e and A_e are unitless as Q_s and Q_a both have the units of Watts, and the Poynting vector $\vec{S^i}$ has the units of Watts per square meter, and the apparent projected size, π times the square of the radius, has the units of square meters.

These quantities can all be computed systematically just with a knowledge of the expansion coefficients of the scattered radiation and the expansion coefficients of spherical harmonic representation of the plane wave representing the impinging electromagnetic

wave. Suppose that $\vec{E}^s$ and $\vec{H}^s$ are the electric and magnetic vectors of the scattered radiation and suppose that $\vec{E}^i$ and $\vec{H}^i$ are the electric and magnetic vectors of the incoming radiation that stimulates the sphere filled with electromagnetic material. The quantity

$$Q_s + Q_a = \text{the total extinguished power} \tag{3.1.7}$$

is called the *extinction* and is calculated by integrating the Poynting vector,

$$\vec{S} = (1/2)(\vec{E}^s + \vec{E}^i) \times (\vec{H}^s + \vec{H}^i)^*) \tag{3.1.8}$$

over the outer surface of the sphere. For a plane wave, the result of integrating

$$\vec{S}^i = (1/2)(\vec{E}^i) \times (\vec{H}^i)^*) \tag{3.1.9}$$

over the surface of a sphere is zero, since the average value of the normal vector to this surface is zero. The rate at which energy leaves the surface of the sphere as a result of reradiation of the energy incident on it is similarly determined by integrating

$$\vec{S}^s = (1/2)(\vec{E}^s) \times (\vec{H}^s)^*) \tag{3.1.10}$$

over the surface of the sphere.

In this section we shall study how absorption and optical efficiency depend on the wavelength or the frequency of the incoming radiation, but we shall transform this wavelength or frequency, respectively, into a unitless quantity called the size parameter. If

$$\omega = 2 \cdot \pi \cdot f \tag{3.1.11}$$

then the size parameter is defined as

$$s_\lambda = \frac{2 \cdot \pi \cdot R_N}{\lambda} \tag{3.1.12}$$

where

$$2 \cdot \pi \cdot \lambda^{-1} = \omega \cdot \sqrt{(\mu_0 \epsilon_0)} = 2 \cdot \pi \cdot f \sqrt{(\mu_0 \epsilon_0)} \tag{3.1.13}$$

or

$$\lambda = \frac{2\pi}{(\omega\sqrt{\mu_0 \epsilon_0})} \tag{3.1.14}$$

where

$$\mu_0 = 4\pi \times 10^{-7} \tag{3.1.15}$$

and

$$\epsilon_0 = 8.854 \times 10^{-12} \tag{3.1.16}$$

are the free space magnetic permeability and electrical permittivity.

3.2 Computer Calculations

If we look at the representation of expansion coefficients in terms of index of refraction, we find that as this index of refraction gets close to an imagninary part of $\sqrt{2}$ and a real part near zero, that there is very strong scattering and absorption at apparently periodic values of the size parameter. The first graph below shows the absorption efficiency of a spherical particle with an index of refraction m given by

$$m = .0001 + i(1.4140) \tag{3.2.1}$$

and the subsequent graph shows the optical efficiency for the same index of refraction.

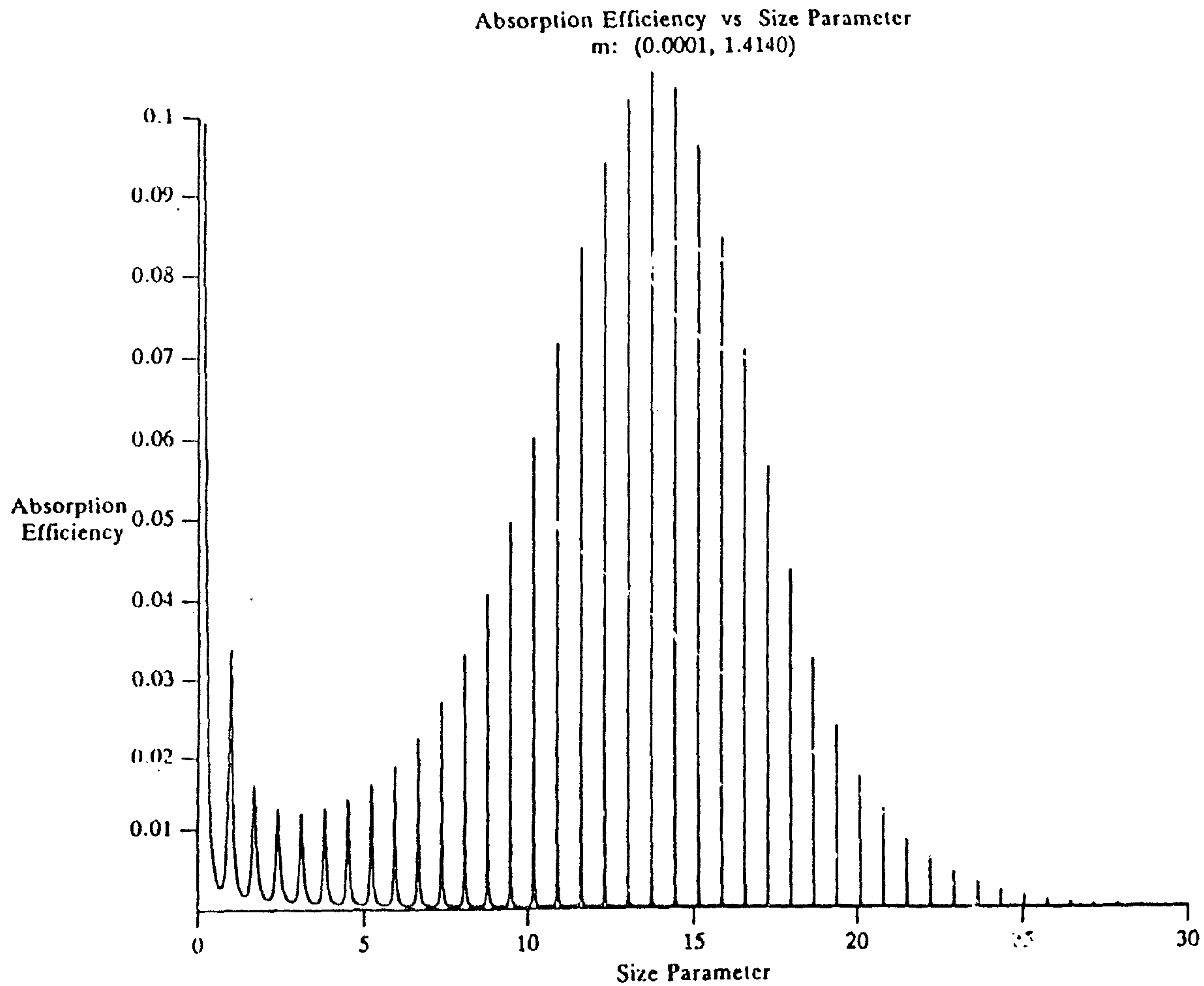


Figure 3.2.1. This graph shows the total absorbed power divided by the product of the length of the Poynting vector of the incident radiation times pi times the square of the radius of the sphere.

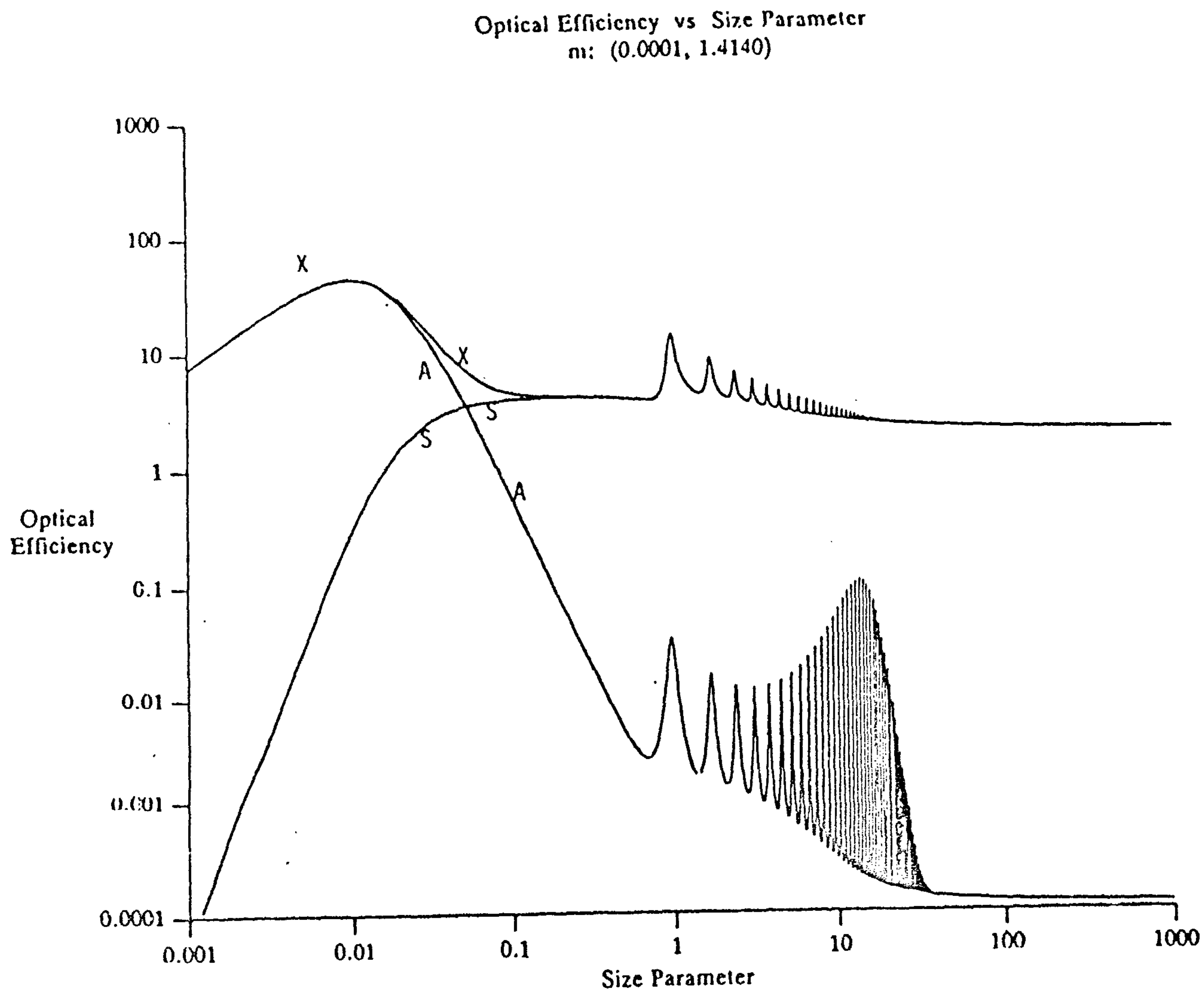


Figure 3.2.2. This graph shows the extinction, X, absorption, denoted by A, and scattering, S, efficiency as a function of the size parameter of a one layer sphere with an index of refraction of .0001 + 1.414i.

These graphs suggest that scattering is much more important than absorption, but as we allow the size parameters to become very large there is a cross over in the scattering and absorption efficiency curves for the same index of refraction. This is shown in the next computation represented on a logarithmic scale which considers size parameters as large as 1000. In this graph, there are the same early maximums as before, but they simply cannot be seen on the logarithmic scale. Some of the maximums are shown in the following table

size parameter	absorption efficiency	optical efficiency
$.010000E0$	$.43077010E+2$	$.3090E+6$
$.973500E0$	$.13852698E+2$	$.2388E+4$
$.168930E1$	$.86167641E+1$	$.9345E+3$
$.239210E1$	$.66955849E+1$	$.6547E+3$
$.309430E1$	$.56436135E+1$	$.6412E+3$
$.379800E1$	$.49728349E+1$	$.7194E+3$
$.450370E1$	$.45070065E+1$	$.8206E+3$

On the vertical axis of these graphs we are computing the logarithm of the efficiencies. When the imaginary part of the index of refraction is slightly above the square root of two, we see a strong peak in optical efficiency that is due to absorption efficiency. The graphs which follow show, over a small range of size parameteres, results for an index of refraction of

$$m = .0001 + i(1.4144) \tag{3.2.2}$$

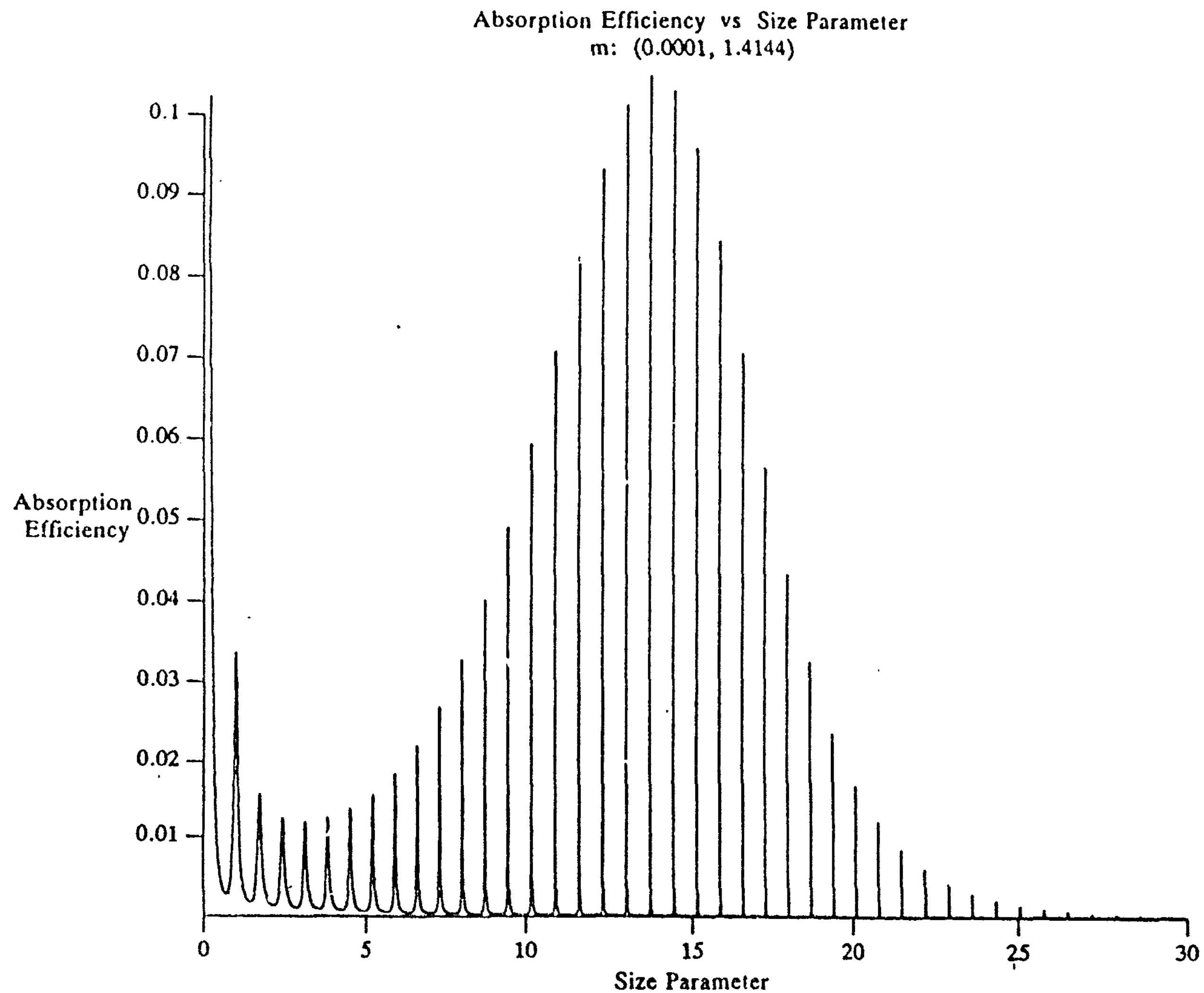


Figure 3.2.3. This graph shows the absorption efficiency, defined in section 3.1, as a function of the size parameter of a one layer sphere with an index of refraction of .0001 + i(1.4144)

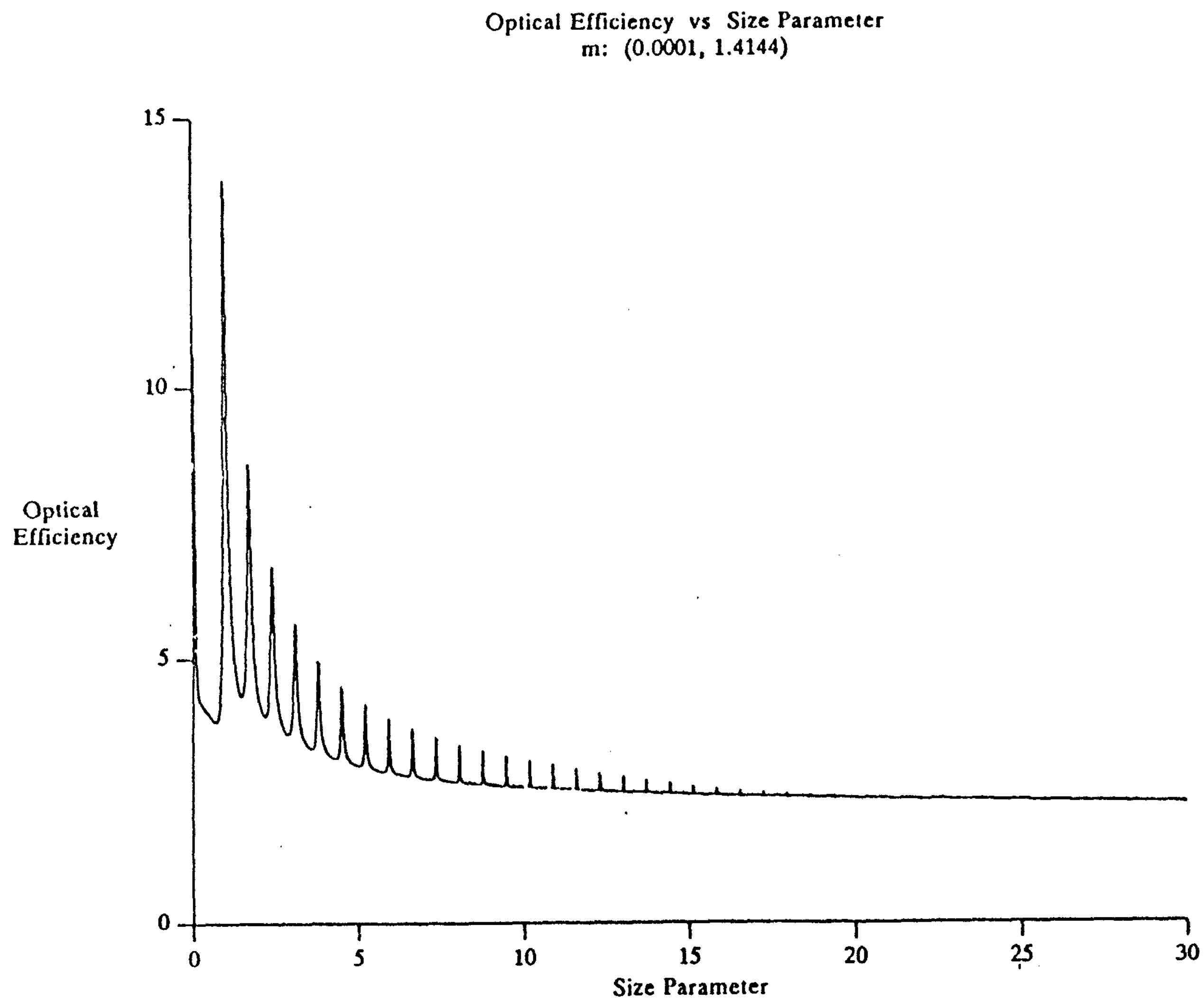


Figure 3.2.4. This graph shows the optical efficiency of a one layer sphere as a function of size parameter when the sphere has an index of refraction of .0001 + i(1.4144).

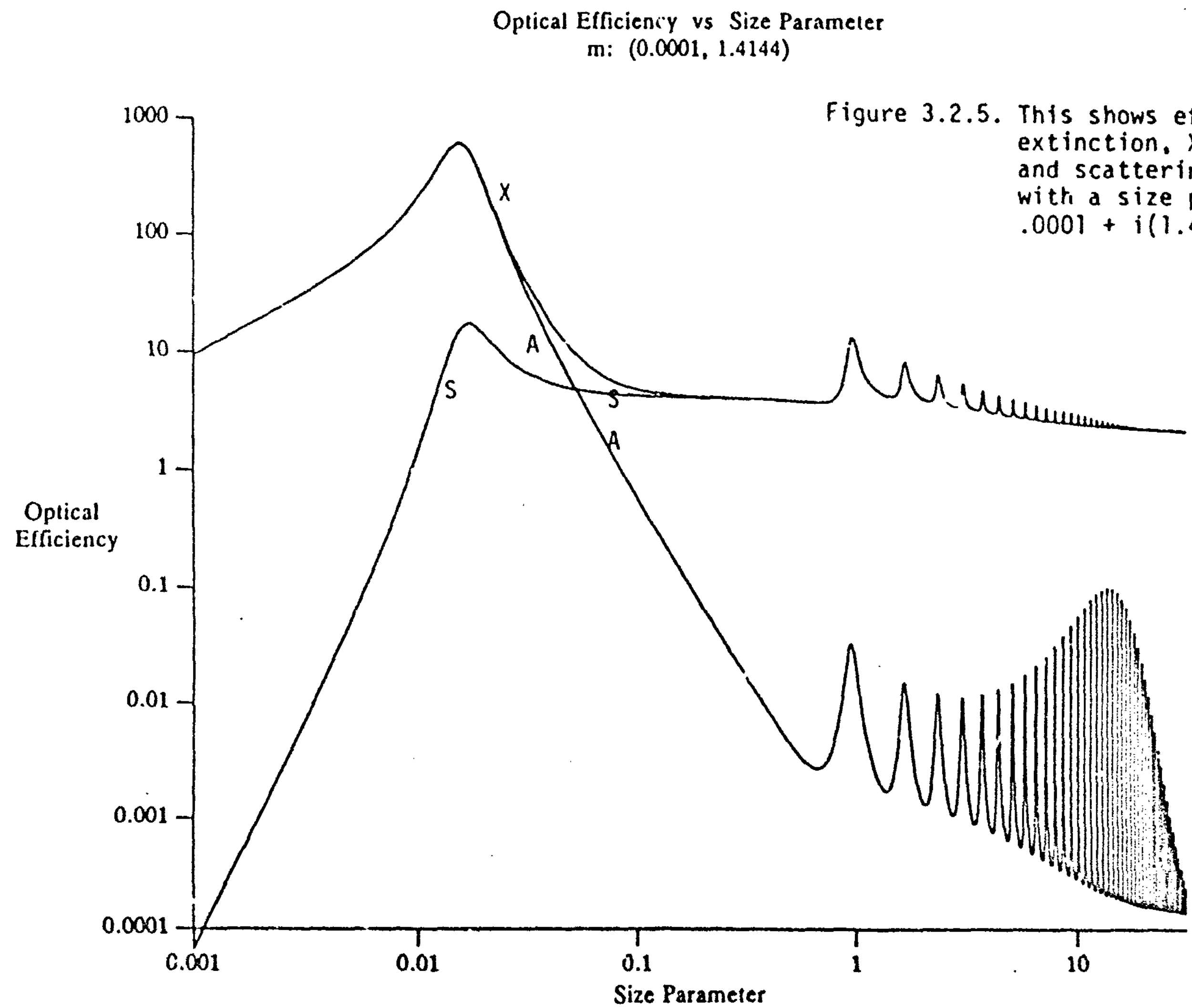


Figure 3.2.5. This shows efficiencies of extinction, X, absorption, A, and scattering, S, for a sphere with a size parameter of .0001 + i(1.4144).

The topmost curve represents extinction, which is scattering plus absorption efficiency. To the left of the size parameter where we see a crossing of curves in the narrow region where we can see three curves, they are from top to bottom, (i) extinction, (ii) absorption, and (iii) scattering efficiency. In the region to the right of the crossover point, where we can see three curves, they are from top to bottom (i) extinction (or optical), (ii) scattering, and (iii) absorption efficiency.

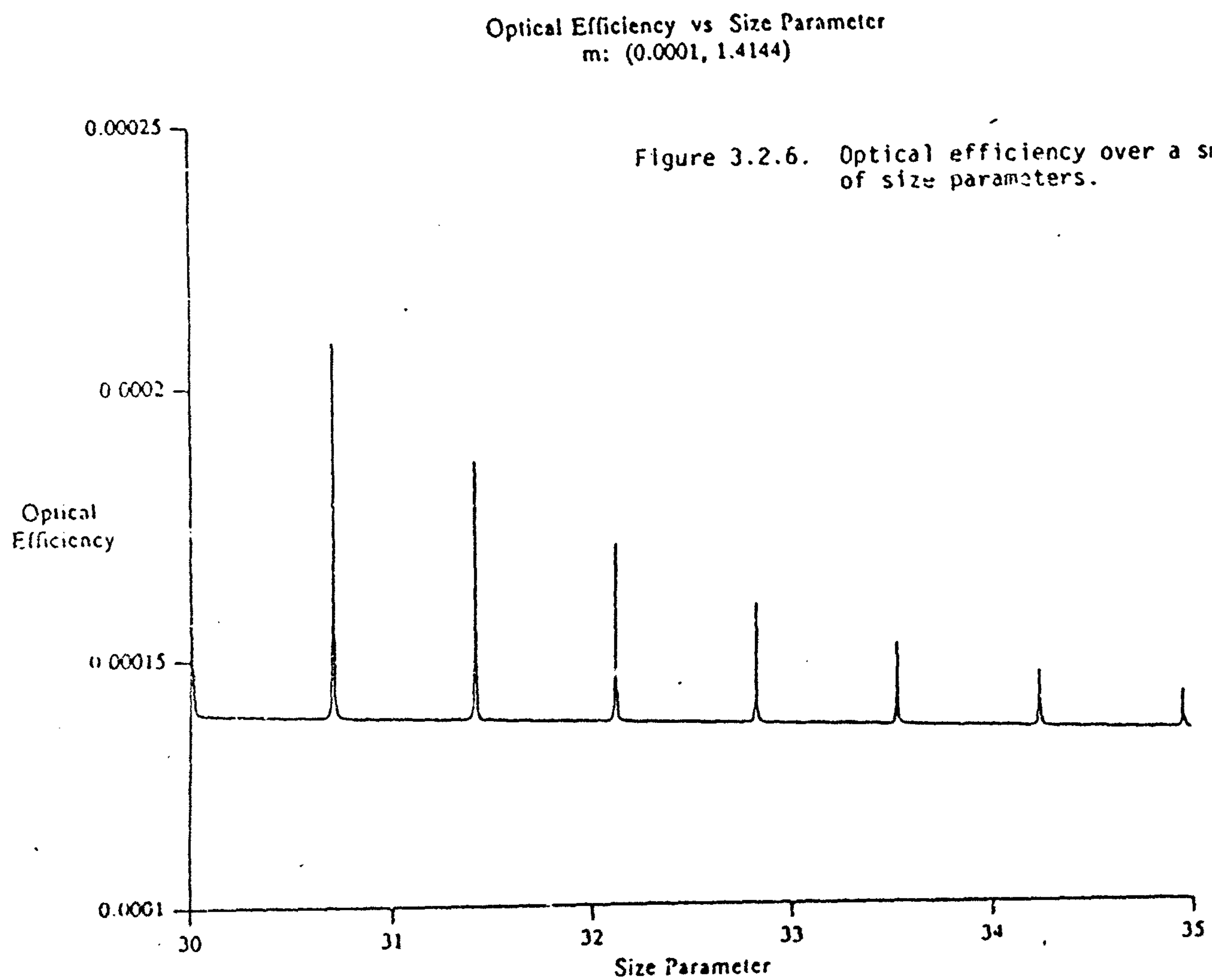


Figure 3.2.6. Optical efficiency over a small range of size parameters.

In these calculations going out to large size parameters, around 8000 size parameters were considered along with a procedure which searched for the maximums and the troughs in the graph. The following table shows a computation of the low points in the graph of absorption efficiency.

size parameter	absorption efficiency
.700	$.38222365E + 1$
.147	$.43105236E + 1$
.218	$.39434158E + 1$
.289	$.35937823E + 1$
.360	$.33305300E + 1$
.431	$.33305300E + 1$
.502	$.29903701E + 1$

Note that the locations of the troughs in the above table are in between the maximums indicated in the previous table. For this particular calculation great care must be exercised in locating the maximums. When the spherical particle has a greater permittivity, the peaks are broader and can generally be observed graphically by a straightforward computation with evenly spaced size parameters.

Absorption Efficiency as a function of size parameter
Avg Frequency = 1000 Megahertz, Radius = 5.0 centimeters
Total absorbed power = 1.80313536E-5 Watts
ϵ = 59.86
μ radial = 2+0.1i, and μ tangential = 4+0.2i
σ = 1.00152 mhos per meter
Constant index of refraction = (15.59143572,2.69320705)

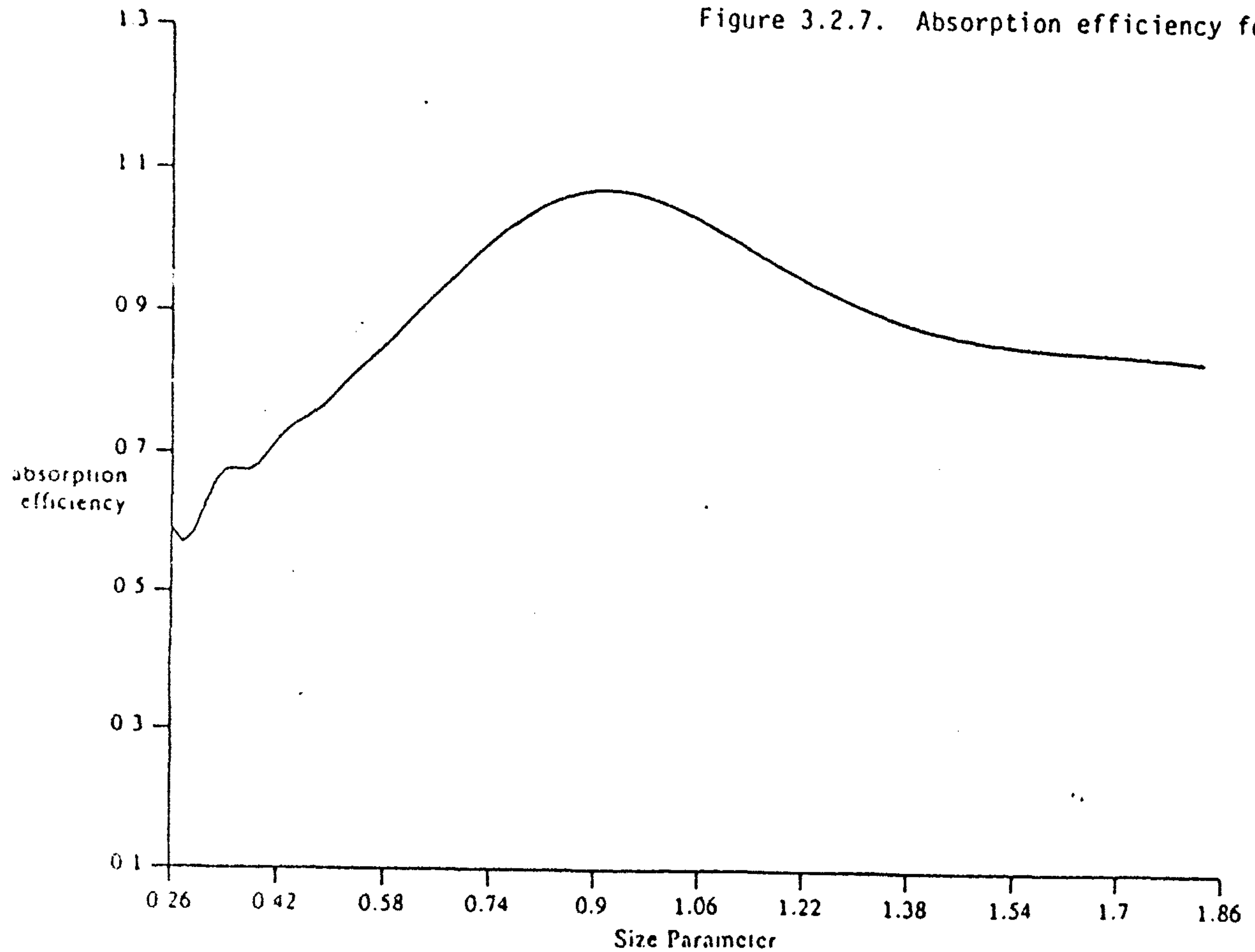


Figure 3.2.7. Absorption efficiency for a lossy sphere.

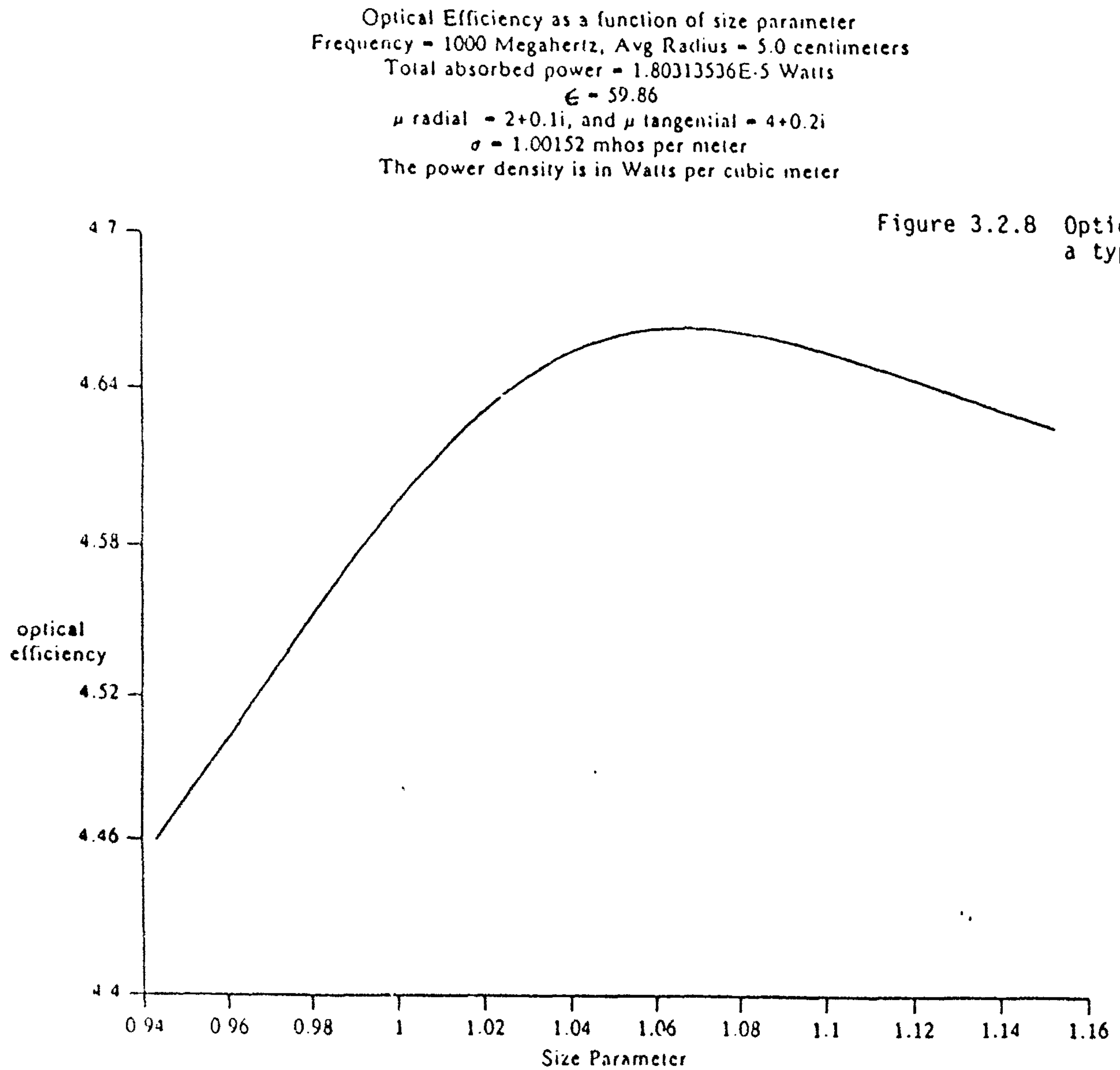


Figure 3.2.8 Optical efficiency for a typical lossy sphere.

3.3 Highly Efficient Two Layer Spheres

Van de Hulst ([47]) develops the relation between the permittivity ϵ_1 of the core of radius qR, where q is a number between zero and one and the permittivity ϵ_2 of the shell of outer radius R which will produce a very high efficiency. This relationship ([47]) is

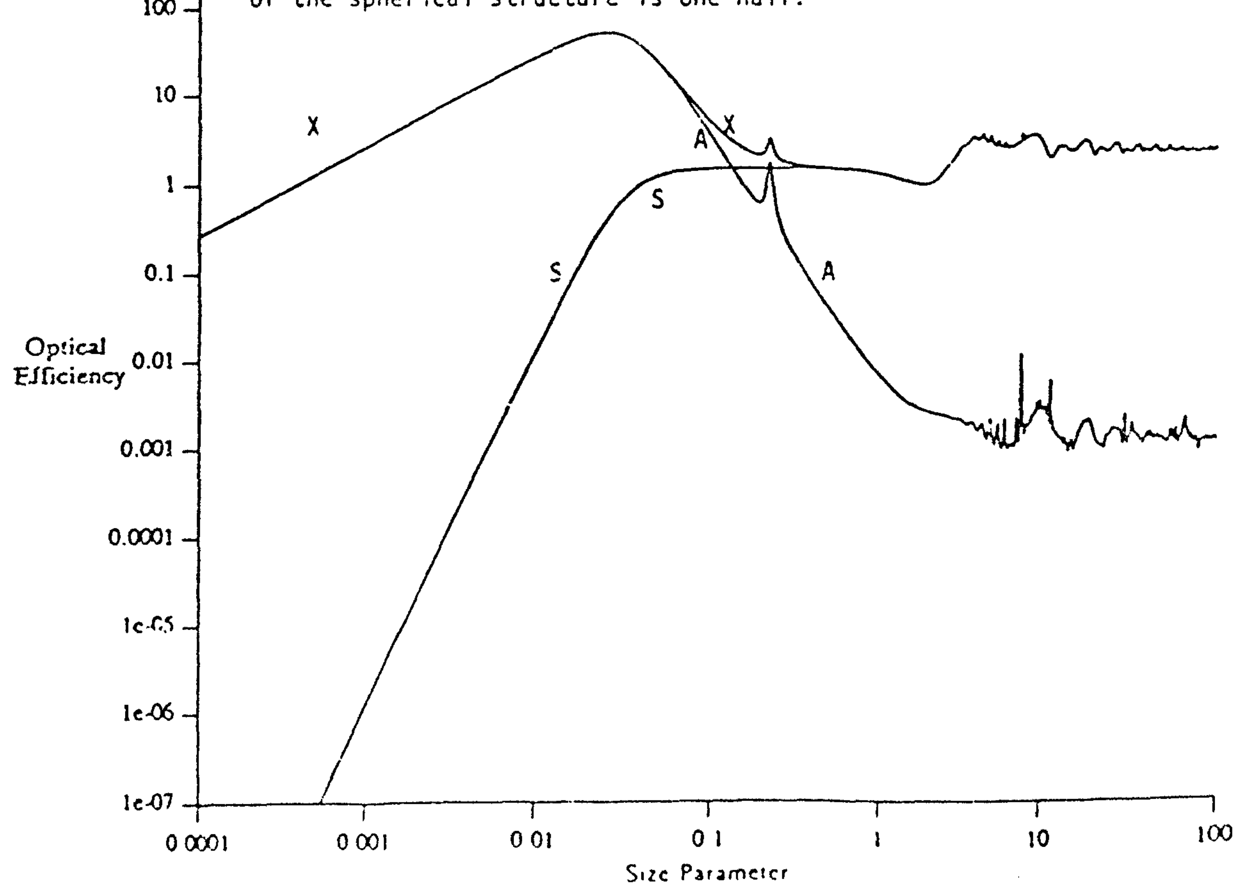
$$\epsilon_1 = \left(\frac{(1 - 2q^3)\epsilon_2^2 + \epsilon_2(4 + 2q^3)}{(2 - 2q^3) + \epsilon_2(1 + 2q^3)} \right) \tag{3.3.1}$$

The following shows some computations of efficiency for two layer structures which nearly satisfy this relationship.

Figure 3.3.1

Optical Efficiency vs Size Parameter
ms: (2, 0) mc:(0.001, 2) vc/v: 0/5

The figure below shows extinction, X, absorption, A, and scattering, S, efficiency as a function of size parameter. The index of refraction of the shell is 2 + 0i and the index of refraction of the core is .001 + 2i. The ratio of the volume of the core to the total volume of the spherical structure is one half.



Backscattering vs Size Parameter
ms: (2, 0) mc: (0, 2),(1.e-6, 2) vc/v: 0.5

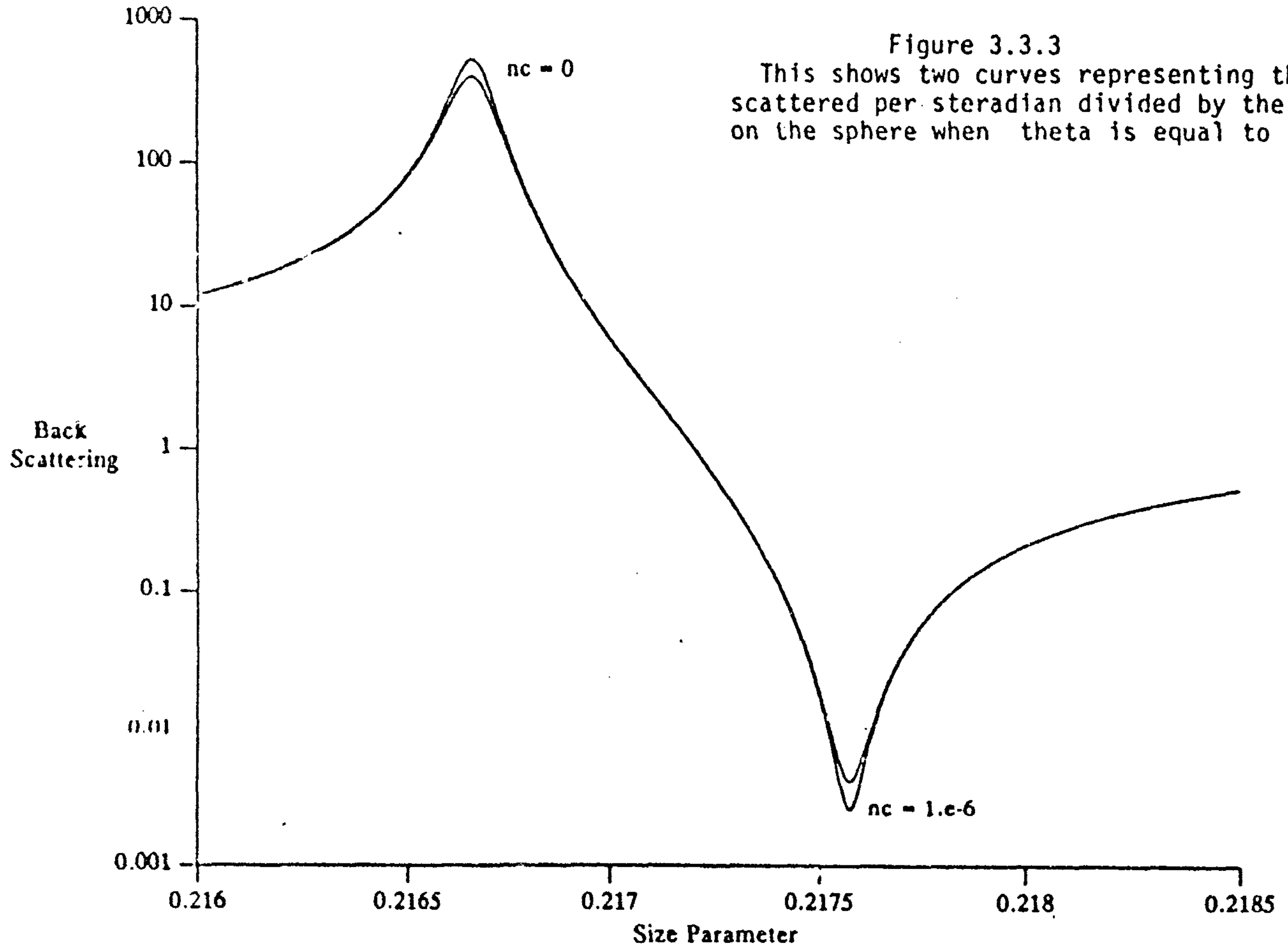


Figure 3.3.3
This shows two curves representing the total power scattered per steradian divided by the power incident on on the sphere when theta is equal to 180 degrees.

This figure shows backscattered power versus size parameter for a pair of two layer spheres. The index of refraction of the shell is 2+0i, and the index of the inner core is either 2i or .000001 + 2i, and are marked above as nc = 0 or nc = 1.e-6.

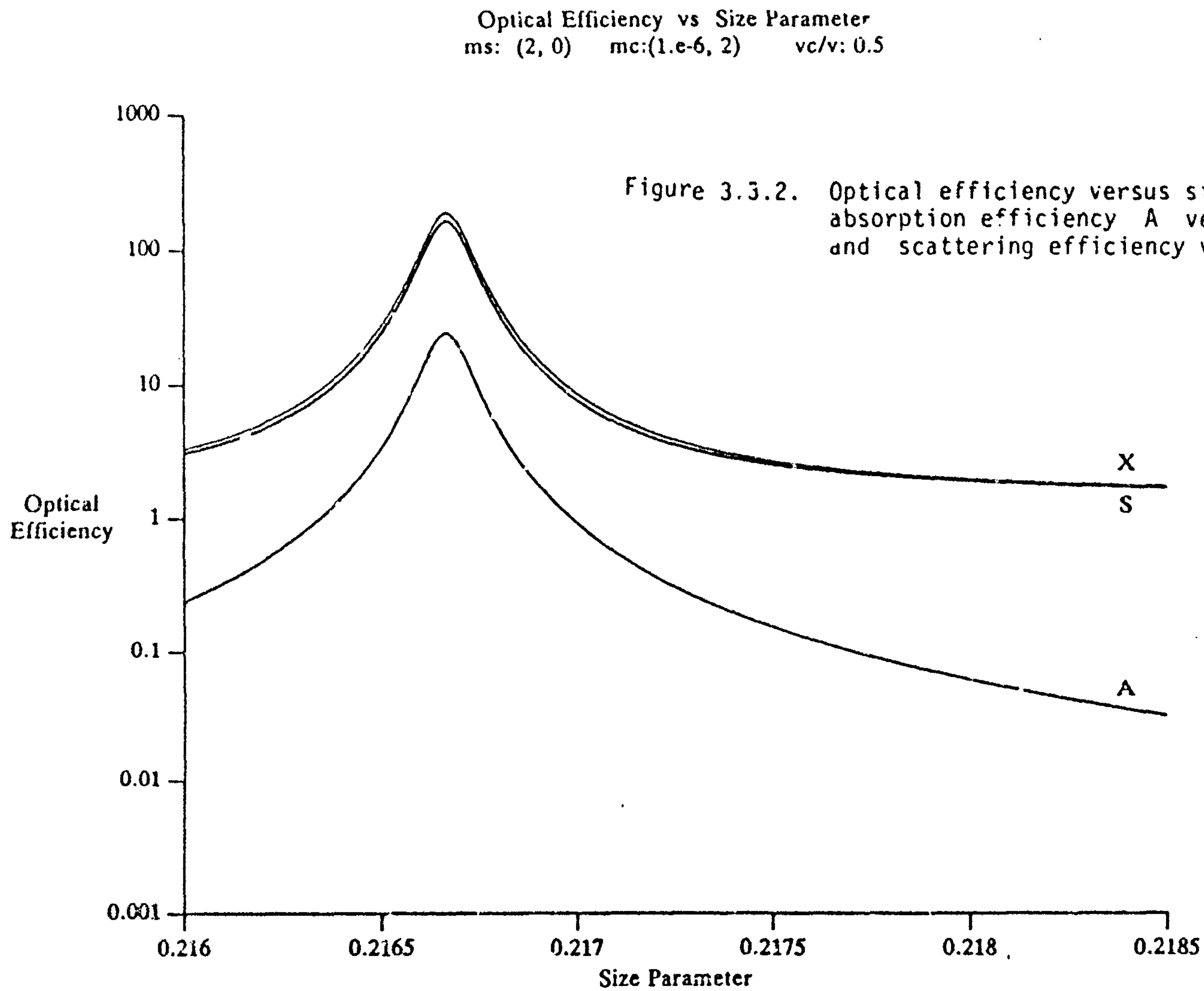


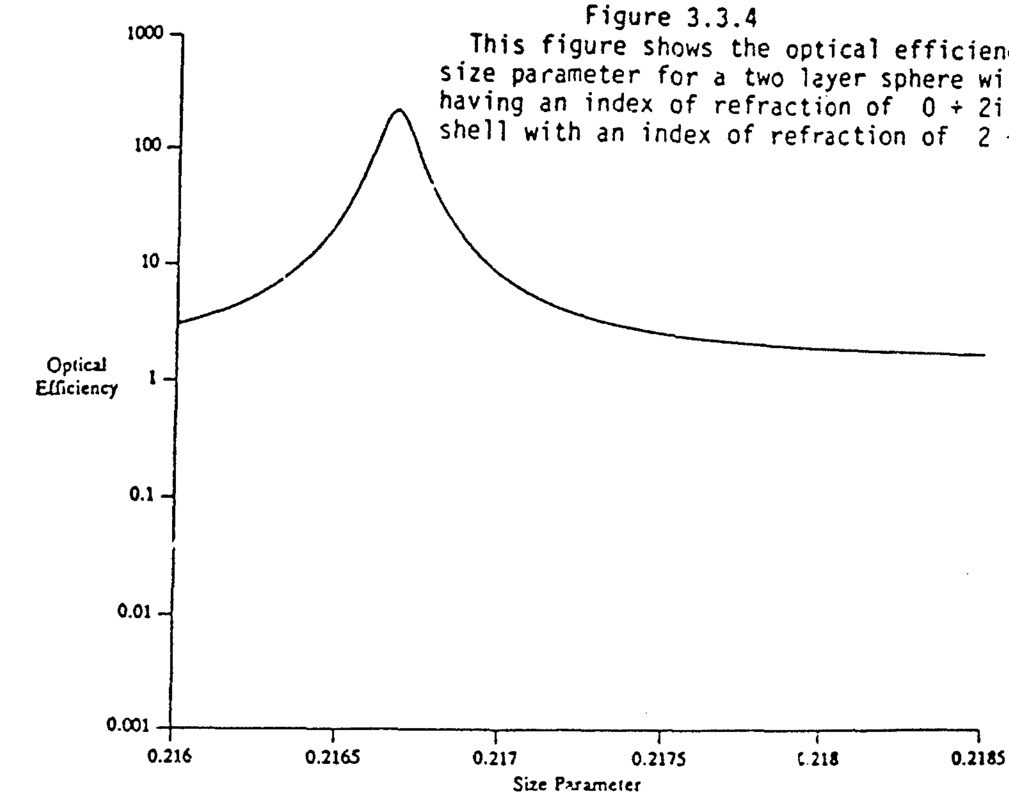
Figure 3.3.2. Optical efficiency versus size parameter, absorption efficiency A versus size parameter, and scattering efficiency versus size parameter.

In the above figure X represents extinction, S scattering, and A absorption efficiency. The ratio of the volume of the core to the total volume of the sphere is .5

Optical Efficiency vs Size Parameter
ms: (2, 0) mc:(0, 2) vc/v: 0.5

Figure 3.3.4

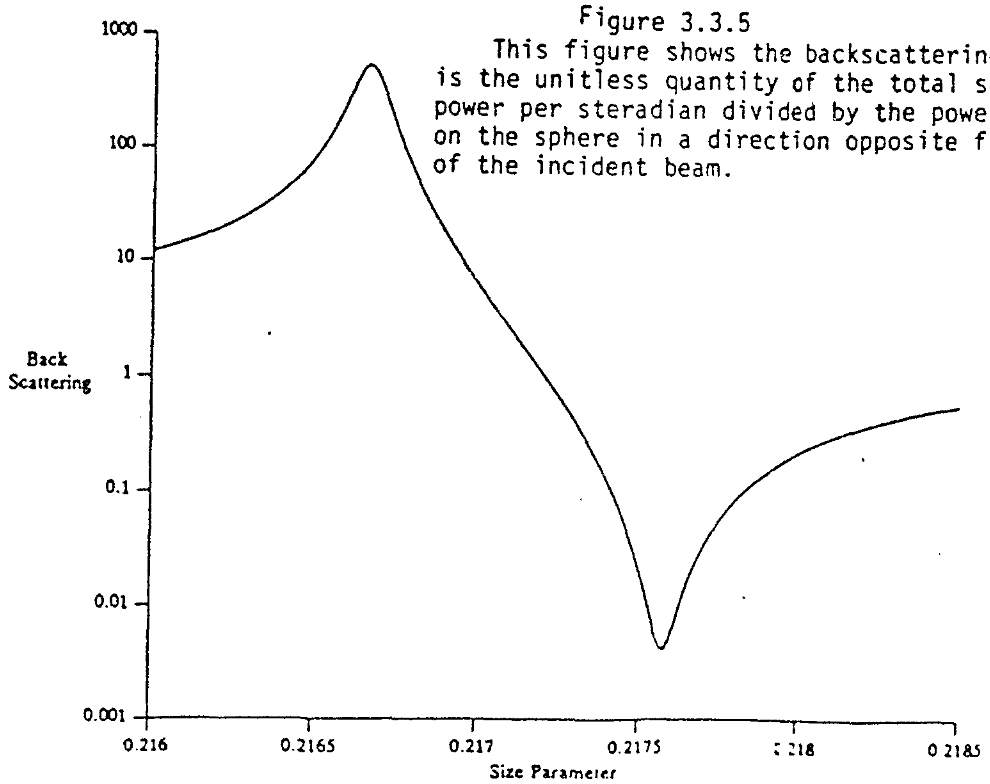
This figure shows the optical efficiency versus size parameter for a two layer sphere with a core having an index of refraction of 0 + 2i and a shell with an index of refraction of 2 + 0i.



Backscattering vs Size Parameter
ms: (2, 0) mc:(0, 2) vc/v: 0.5

Figure 3.3.5

This figure shows the backscattering which is the unitless quantity of the total scattered power per steradian divided by the power incident on the sphere in a direction opposite from that of the incident beam.



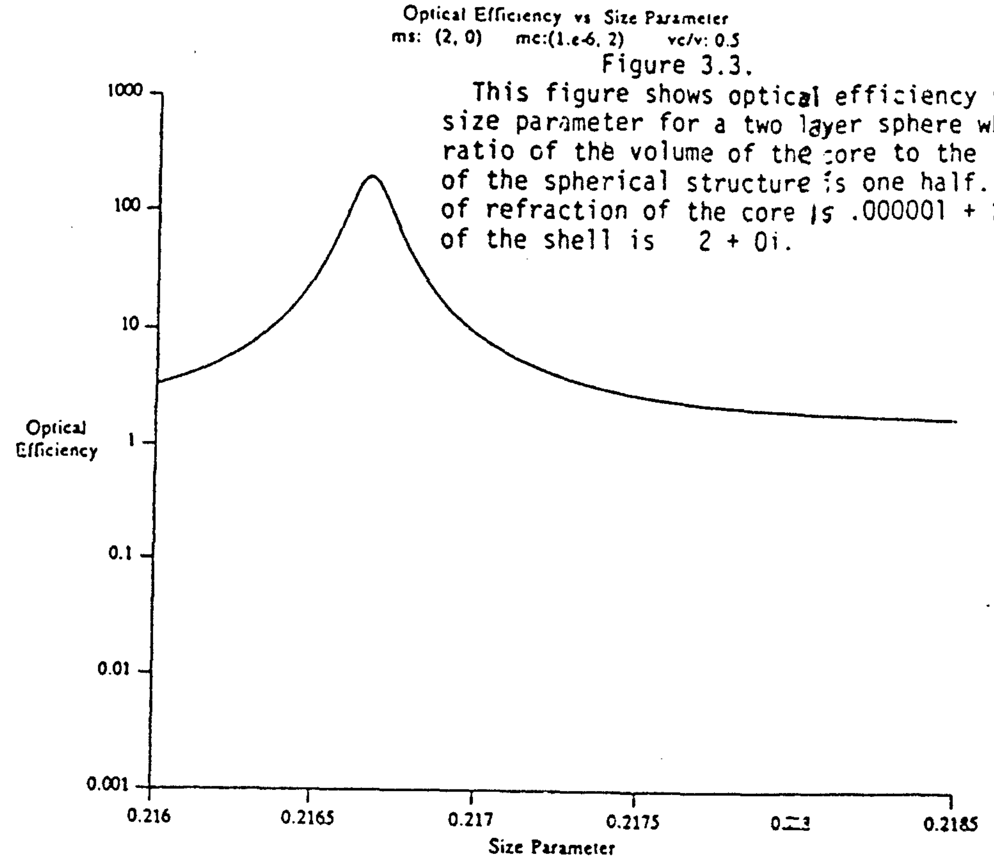


Figure 3.3.
This figure shows optical efficiency versus size parameter for a two layer sphere where the ratio of the volume of the core to the total volume of the spherical structure is one half. The index of refraction of the core is .000001 + 2i and that of the shell is 2 + 0i.

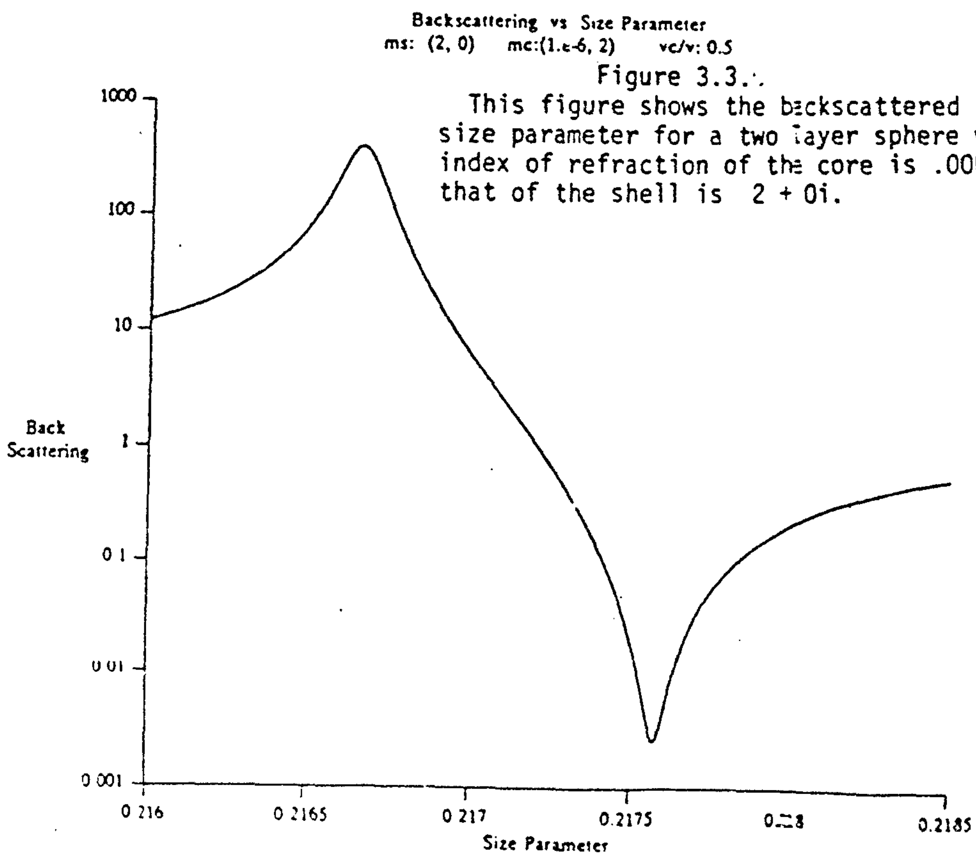


Figure 3.3.
This figure shows the backscattered power versus size parameter for a two layer sphere where the index of refraction of the core is .000001 + 2i and that of the shell is 2 + 0i.

4 Spatially Complex Sources

4.1 Expansion Coefficient Determination

We provide the user with an analysis of the response of an N layer structure to spatially and temporally complex sources of electromagnetic radiaton. Let $\vec{E}(x, y, z, t)$ and $\vec{H}(x, y, z, t)$ be the electric and magnetic fields of a complex source with Fourier transforms $\hat{\vec{E}}(x, y, z, \omega)$ and $\hat{\vec{H}}(x, y, z)$. We suppose that this radiation source exists in layer

$$p \in \{2, 3, \cdots, N+1\}.$$

where N is the number of layers in the spherical stucture. Let us sup... ...energy source in layer p has an electric vector (see equation 2.1.1) given by

$$\vec{E} = \sum_{(m,n)\in\mathcal{I}} \Big\{ \tilde{a}_{(m,n)} Z^{(a,1)}_{(n,p)}(r) \vec{A}_{(m,n)}(\theta, \phi) +$$

$$[-n(n+1)\{\zeta_b^{(p)}\}]\tilde{b}_{(m,n)} \frac{Z^{(b,1)}_{(n,p)}(r)}{kr} \vec{C}_{(m,n)}(\theta, \phi) +$$

$$\frac{\tilde{b}_{(m,n)}}{k_p r} \left(- \left(\frac{\partial}{\partial r} \right) \left(r Z^{(b,1)}_{(n,p)}(r) \right) \right) \vec{B}_{(m,n)}(\theta, \phi) \Big\} \tag{4.1.1}$$

Observe that the coefficients $\tilde{a}^{(p)}_{(m,n)}$ are determined for every $p > 1$ by the relation,

$$\underset{r \to R_{p-1}}{LIM} \left[\frac{\int\int_{C(r)} \hat{\vec{E}}_p(x, y, z, \omega) \cdot \vec{A}_{(m,n)}(\theta, \phi)^* sin(\theta) d\theta d\phi}{\int\int_{C(r)} \vec{A}_{(m,n)}(\theta, \phi) \cdot \vec{A}_{(m,n)}(\theta, \phi)^* sin(\theta) d\theta d\phi} \right]$$

$$= \tilde{a}^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p)}(R_{p-1}) \tag{4.1.2}$$

where

$$C(r) = \left\{ (x, y, z) : x^2 + y^2 + z^2 = r^2 \right\} \tag{4.1.3}$$

Thus, equation (4.1.2) gives us the expansion coefficients for the representation of $\vec{E}$ just outside the sphere $C(R_{p-1})$ defined by equation (4.1.3). The coefficients $\tilde{b}^{(p)}_{(m,n)}$ are determined by the equation,

$$\underset{r \to R_{p-1}}{LIM} \left[\frac{\int\int_{C(r)} \hat{\vec{E}}_p(x, y, z, \omega) \cdot \vec{B}_{(m,n)}(\theta, \phi)^* sin(\theta) d\theta d\phi}{\int\int_{C(r)} \vec{B}_{(m,n)}(\theta, \phi) \cdot \vec{B}_{(m,n)}(\theta, \phi)^* sin(\theta) d\theta d\phi} \right]$$

$$= \tilde{b}^{(p)}_{(m,n)}(-W^{(b,1)}_{(n,p)}(R_{p-1})) \tag{4.1.4}$$

where, using the definition (see equation 2.1.12),

$$W^{(b,j)}_{(n,p)}(r) = \frac{1}{k_p r}\left(\frac{\partial}{\partial r}\right)(rZ^{(b,j)}_{(n,p)}(r) \tag{4.1.5}$$

and the functions $\vec{A}_{(m,n)}(\theta,\phi)$ and $\vec{B}_{(m,n)}(\theta,\phi)$ are given by equations (1.3.1) and (1.3.2). We will show that the integrals in the denominators in equations (4.1.2) and (4.1.4) can be determined by an exact formula. To exactly evaluate the integrals appearing in the denominators, we use the equation (see Bell [10], equation 11 and equation 18) which states that

$$\int_{-\pi}^{\pi}\int_{C}^{\pi}\left\{\left(\frac{d}{d\theta}P_n^m(cos(\theta))\right)^2 + m^2\frac{P_n^m(cos(\theta))^2}{sin^2(\theta)}\right\}sin(\theta)d\theta d\phi =$$

$$= \left(\frac{2}{2n+1}\right)\left(\frac{(n+m)!}{(n-m)!}\right)n(n+1) \tag{4.1.6}$$

where the functions $P_n^m(x)$ are defined by

$$P_n^m(x) = \frac{(1-x^2)^{m/2}}{2^n n!}D^{n+m}(x^2-1)^n \tag{4.1.7}$$

of the associated Legendre function.

We use the basic definition

$$P_n^m(x) = \left(\frac{(1-x^2)^{m/2}}{2^n n!}\right)D^{n+m}(x^2-1)^n \tag{4.1.8}$$

of the associated Legendre function. If

$$x = cos(\theta) \tag{4.1.9}$$

then

$$d\theta = -\frac{dx}{\sqrt{1-x^2}} \tag{4.1.10}$$

and

$$\int_0^{\pi} P_n^m(cos(\theta))^2 sin(\theta)d\theta =$$

$$\frac{1}{2^{2n}(n!)^2}\int_{-1}^{+1}(1-x^2)^m(D^{n+m}(x^2-1)^n)^2dx = \frac{2(n+m)!}{(2n+1)(n-m)!} \tag{4.1.11}$$

The orthogonality relationship follows from the fact that

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -sin(\theta)\frac{d}{dx} \tag{4.1.12}$$

implies that

$$\begin{aligned} A^{m}_{(n,r)} &= \int_0^{\pi} \left[\frac{d}{d\theta}P_n^m(cos(\theta))\right]\left[\frac{d}{d\theta}P_r^m(cos(\theta))\right] sin(\theta)d\theta \\ &= \int_{-1}^{1}(1-x^2)\frac{d}{dx}P_n^m(x)\frac{d}{dx}P_r^m(x)dx \end{aligned} \tag{4.1.13}$$

The derived identity then follows from an integration by parts and a use of the differential equation relationship,

$$\begin{aligned} &(1-x^2)\left[\left(\frac{d}{dx}\right)^2 P_n^m(x)\right] + (-2x)\frac{d}{dx}P_n^m(x) = \\ &\left[-n(n+1) + \frac{m}{1-x^2}\right] P_n^m(x) \end{aligned} \tag{4.1.14}$$

Details of the analysis can be found in ([11]) and the basic properties of P_n^m are found in ([52])

4.2 An Exterior Complex Source

We now define intralayer relationships that give us the induced field when there are no sources in layers indexed by

$$p \in \{2, 3, \cdots, N\}$$

where N is the number of layers in the sphere. The intralayer relationship yields, for a penetrable core,

$$\begin{bmatrix} a^{(1)}_{(m,n)} \\ 0 \\ b^{(1)}_{(m,n)} \\ 0 \end{bmatrix} = S_N \begin{bmatrix} \bar{a}^{(N+1)}_{(m,n)} \\ \alpha^{(N+1)}_{(m,n)} \\ \bar{b}^{(N+1)}_{(m,n)} \\ \beta^{(N+1)}_{(m,n)} \end{bmatrix} \tag{4.2.1}$$

We can separate the four, a priori unknown coefficients, from the known expansion coefficients of the known external source by rewriting equation (4.2.1) in the form

$$\begin{bmatrix} a^{(1)}_{(m,n)} \\ 0 \\ b^{(1)}_{(m,n)} \\ 0 \end{bmatrix} - \mathcal{S}_N \begin{bmatrix} 0 \\ \alpha^{(N+1)}_{(m,n)} \\ 0 \\ \beta^{(N+1)}_{(m,n)} \end{bmatrix} = \mathcal{S}_N \begin{bmatrix} \tilde{a}^{(N+1)}_{(m,n)} \\ 0 \\ \tilde{b}^{(N+1)}_{(m,n)} \\ 0 \end{bmatrix} \tag{4.2.2}$$

Thus, relating the a priori unknown coefficients to the known expansion coefficients $\tilde{a}^{(N+1)}_{(m,n)}$ and $\tilde{b}^{(N+1)}_{(m,n)}$ reduces to the problem of finding the inverse of the matrix

$$\mathcal{T} = \overline{\overline{I}} - \mathcal{S}_N \tag{4.2.3}$$

4.3 Interior Sources

We now suppose that there are interior sources in the layers. This could be important in assessing the impact of a sweeping radar on a person living near the radar who has one or more metalic implants to replace broken bones or clamps to hold them in place. The potentially serious nature of this can be seen from the fact that ([55] p 40) has used this concept to postulate a design for an electromagnetic missile.

With interior sources, the expansion coefficients in the free space surrounding the N layer sphere and the expansion coefficients in the inner core will be shown to be related by affine transformations rather than linear transformations.

We model complex sources in a layer by allowing an arbitrary representation of a source in terms of an expansion in a Hilbert space of vector valued functions. We assume that if the shell containing the source lies between $r = R_p$ and $r = R_{p+1}$ and represent the expansion coefficients of the electric field due to this source in the inner shell in terms of expansion coefficients $\tilde{a}^{(p)}_{(m,n)}$, $\tilde{b}^{(p)}_{(m,n)}$. We assume that these are given and represent a source located at a point $r = \tilde{R}_p$ that is between $r = R_p$ and $r = R_{p+1}$. These are obtained by assuming that the source is unaffected by the medium and that the currents, say in a dipole source, are used to represent an electric vector $\tilde{\vec{E}}_p$. This electric vector is then

represented on the inner shell by the relations,

$$\underset{r \to R_p}{LIM} \left[\frac{\int\int_{C(r)} \tilde{\vec{E}}_p(x,y,z,\omega) \cdot \vec{A}_{(m,n)}(\theta,\phi)^* sin(\theta) d\theta d\phi}{\int\int_{C(r)} \vec{A}_{(m,n)}(\theta,\phi) \cdot \vec{A}_{(m,n)}(\theta,\phi)^* sin(\theta) d\theta d\phi} \right]$$

$$= \tilde{a}^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p)}(k_p R_p) \tag{4.3.1}$$

The values of the expansion coefficients $\tilde{b}^{(p)}_{(m,n)}$ of this source field on the shell $r = R_p$ are given by

$$\underset{r \to R_p}{LIM} \left[\frac{\int\int_{C(r)} \tilde{\vec{E}}_p(x,y,z,\omega) \cdot \vec{B}_{(m,n)}(\theta,\phi)^* sin(\theta) d\theta d\phi}{\int\int_{C(r)} \vec{B}_{(m,n)}(\theta,\phi) \cdot \vec{B}_{(m,n)}(\theta,\phi)^* sin(\theta) d\theta d\phi} \right]$$

$$= \tilde{b}^{(p)}_{(m,n)}(-W^{(b,1)}_{(n,p)}(k_p R_p)) \tag{4.3.2}$$

Thus, we know the electric field due to the isolated source at this point on the shell $r = R_p$. However, unlike the source in the space surrounding the N layer spherical structure we cannot assume that the field is represented by these expansion coefficients and the expansion coefficients $\alpha^{(p)}_{(m,n)}$ and $\beta^{(p)}_{(m,n)}$ used to represent the radiation emanating from the inner shell, as there may be additional sources coming from beyond $r = R_p$ that are due to external sources and reflections of these sources from the layer $r = R_p$. Instead we approximate the representation of this source by a finite linear combination of vector spherical wave functions and assume that at some point $r = \tilde{R}_p$ possibly just slightly smaller than the location of the actual source, so that value of the field at the point considered would not be singular, we impose essentially an impedance boundary condition (Wu [55]) at $r = R_p$ which will give us a relationship between the general expansion coefficients $\alpha^{(p,-)}_{(m,n)}$ and $\beta^{(p,-)}_{(m,n)}$ and $a^{(p,-)}_{(m,n)}$ and $b^{(p,-)}_{(m,n)}$ used to represent the fields when $r < \tilde{R}_p$ and the expansion coefficients $\alpha^{(p,+)}_{(m,n)}$ and $\beta^{(p,+)}_{(m,n)}$ and $a^{(p,+)}_{(m,n)}$ and $b^{(p,+)}_{(m,n)}$ that are used to represent the fields when $r > \tilde{R}_p$. We suppose that the magnetic vector just outside $r = \tilde{R}_p$ is denoted by $\vec{H}^+_p$ and that the magnetic vector just inside $r = \tilde{R}_p$ is given by $\vec{H}^-_p$ and that the boundary conditions used to relate the expansion coefficients $\alpha^{(p,-)}_{(m,n)}$ and $\beta^{(p,-)}_{(m,n)}$ and $a^{(p,-)}_{(m,n)}$ and $b^{(p,-)}_{(m,n)}$ for $R_p < r < \tilde{R}_p$ to the expansion coefficients $\alpha^{(p,+)}_{(m,n)}$ and $\beta^{(p,+)}_{(m,n)}$ and $a^{(p,+)}_{(m,n)}$ and $b^{(p,+)}_{(m,n)}$ for $R_{p+1} > r > \tilde{R}_p$ are continuity of tangential components of $\vec{E}$ and the nonhomogeneous impedance boundary condition

$$\vec{n} \times (\vec{H}^+_p - \vec{H}^-_p) = (i\omega\epsilon + \sigma)(\tilde{\vec{E}}_p - (\tilde{\vec{E}}_p \cdot \vec{e}_r)\vec{e}_r) \tag{4.3.3}$$

Taking the dot product of both sides of equation (4.3.3) with respect to the vector

$$\vec{V}_B = \vec{B}_{(m,n)}(\theta, \phi)$$

and integrating over the sphere $r = \tilde{R}_p$ we see that

$$\begin{aligned}
&\frac{i}{\omega\mu^{(p)}} \left\{ a^{(p,+)}_{(m,n)} k_p W^{(a,1)}_{(n,p)}(k_p \tilde{R}_p) + \alpha^{(p,+)}_{(m,n)} k_p W^{(a,3)}_{(n,p)}(k_p \tilde{R}_p) \right\} + \\
&\left(\frac{-i}{\omega\mu^{(p)}} \right) \left\{ \alpha^{(p)} b^{(p,+)}_{(m,n)} (W^{(b,1)}_{(n,p)}(k_p \tilde{R}_p)) + \alpha^{(p)} \beta^{(p,+)}_{(m,n)} (W^{(b,3)}_{(n,p)}(k_p \tilde{R}_p)) \right\} = \\
&\frac{i}{\omega\mu^{(p)}} \left\{ a^{(p,-)}_{(m,n)} k_p W^{(a,1)}_{(n,p)}(k_p \tilde{R}_p) + \alpha^{(p,-)}_{(m,n)} k_p W^{(a,3)}_{(n,p)}(k_p \tilde{R}_p) \right\} + \\
&\left(\frac{-i}{\omega\mu^{(p)}} \right) \left\{ \alpha^{(p)} b^{(p,-)}_{(m,n)} (W^{(b,1)}_{(n,p)}(k_p \tilde{R}_p)) + \alpha^{(p)} \beta^{(p,-)}_{(m,n)} (W^{(b,3)}_{(n,p)}(k_p \tilde{R}_p)) \right\} + \\
&(i\omega\epsilon^{(p)} + \sigma^{(p)})[-\tilde{b}^{(p)}_{(m,n)} W'^{(b,1)}_{(n,p)}(k_p \tilde{R}_p) - \tilde{\beta}^{(p)}_{(m,n)} W^{(b,3)}_{(n,p)}(k_p \tilde{R}_p)]
\end{aligned} \tag{4.3.4}$$

Taking the dot product of both sides of equation (4.3.3) with respect to the vector

$$\vec{V}_A = \vec{A}_{(m,n)}(\theta, \phi)$$

and integrating over the sphere $r = \tilde{R}_p$ we see that

$$\begin{aligned}
&\left(\frac{i}{\omega\mu^{(p)}} \right) [-k_p] \left\{ b^{(p,+)}_{(m,n)} Z^{(b,1)}_{(n,p)}(k_p \tilde{R}_p) + \beta^{(p,+)}_{(m,n)} Z^{(b,3)}_{(n,p)}(k_p \tilde{R}_p) \right\} = \\
&\left(\frac{i}{\omega\mu^{(p)}} \right) \left\{ \alpha^{(p)} a^{(p,+)}_{(m,n)} Z^{(a,1)}_{(n,p)}(k_p \tilde{R}_p) + \alpha^{(p)} \alpha^{(p,+)}_{(m,n)} Z^{(a,3)}_{(n,p)}(k_p \tilde{R}_p) \right\} + \\
&(i\omega\epsilon^{(p)} + \sigma^{(p)})[\tilde{a}^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p)}(k_p \tilde{R}_p) + \tilde{\alpha}^{(p)}_{(m,n)} Z^{(a,3)}_{(n,p)}(k_p \tilde{R}_p)]
\end{aligned} \tag{4.3.5}$$

Using the fact that $\tilde{\alpha}^{(p)}_{(m,n)} = 0$ and $\tilde{\beta}^{(p)}_{(m,n)} = 0$ and that the coefficients $\tilde{a}^{(p)}_{(m,n)}$ and $\tilde{b}^{(p)}_{(m,n)}$ are completely known gives us a simple relationship between the expansion coefficients (see equation 2.1.13)

$$\begin{aligned}
&[a^{(p,+)}_{(m,n)} Z^{(a,1)}_{(n,p)}(k_p \tilde{R}_p) + \alpha^{(p,+)}_{(m,n)} Z^{(a,3)}_{(n,p)}(k_p \tilde{R}_p)] \\
&= [a^{(p,-)}_{(m,n)} Z^{(a,1)}_{(n,p)}(k_p \tilde{R}_p) + \alpha^{(p,-)}_{(m,n)} Z^{(a,3)}_{(n,p)}(k_p \tilde{R}_p)]
\end{aligned} \tag{4.3.6}$$

and mulitplying both sides of the relationship

$$\vec{E}^{+}_{p}(\tilde{R}_p) \cdot \vec{e}_r = \vec{E}^{-}_{p}(\tilde{R}_p) \cdot \vec{e}_r \tag{4.3.7}$$

by $\vec{B}_{(m,n)}(\theta,\phi)$ and integrating over the sphere $r = \tilde{R}_p$ we see that

$$\begin{aligned}
&[-b^{(p,+)}_{(m,n)} W^{(b,1)}_{(n,p)}(k_p\tilde{R}_p) - \beta^{(p,+)}_{(m,n)} W^{(b,3)}_{(n,p)}(k_p\tilde{R}_p)] \\
&= [-b^{(p,-)}_{(m,n)} W^{(b,1)}_{(n,p)}(k_p\tilde{R}_p) - \beta^{(p,-)}_{(m,n)} W^{(b,3)}_{(n,p)}(k_p\tilde{R}_p)] \qquad (4.3.8)
\end{aligned}$$

We define

$$\xi^{(p,2)}_{(m,n)} = 0 = \xi^{(p,4)}_{(m,n)} \qquad (4.3.9)$$

and

$$\xi^{(p,1)}_{(m,n)} = (i\omega\epsilon^{(p)} + \sigma^{(p)})\tilde{a}^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p)}(k_p\tilde{R}_p) \qquad (4.3.10)$$

and finally,

$$(i\omega\epsilon^{(p)} + \sigma^{(p)})\xi^{(p,3)}_{(m,n)} = -\tilde{b}^{(p)}_{(m,n)} W^{(a,1)}_{(n,p)}(\tilde{R}_p) \qquad (4.3.11)$$

The expansion coefficients on opposite sides of the sphere $r = \tilde{R}_p$ are in view of equations (4.3.5), (4.3.4), (4.3.6) and (4.3.8) and equations (4.3.9), (4.3.10), and (4.3.11) are related by

$$\begin{bmatrix} a^{(p,-)}_{(m,n)} \\ \alpha^{(p,-)}_{(m,n)} \\ b^{(p,-)}_{(m,n)} \\ \beta^{(p,-)}_{(m,n)} \end{bmatrix} = \begin{bmatrix} a^{(p,+)}_{(m,n)} \\ \alpha^{(p,+)}_{(m,n)} \\ b^{(p,+)}_{(m,n)} \\ \beta^{(p,+)}_{(m,n)} \end{bmatrix} - S^{-1}_{(n,p)} \begin{bmatrix} \xi^{(p,1)}_{(m,n)} \\ \xi^{(p,2)}_{(m,n)} \\ \xi^{(p,3)}_{(m,n)} \\ \xi^{(p,4)}_{(m,n)} \end{bmatrix} \tag{4.3.12}$$

To complete the determination of the relationship between expansion coefficients in one layer to those in the next one we use equation (2.2.11) and equation (2.2.13) to write

$$\begin{bmatrix} a^{(p,-)}_{(m,n)} \\ \alpha^{(p,-)}_{(m,n)} \\ b^{(p,-)}_{(m,n)} \\ \beta^{(p,-)}_{(m,n)} \end{bmatrix} = Q^{(p)}_n \begin{bmatrix} a^{(p+1,-)}_{(m,n)} \\ \alpha^{(p+1,-)}_{(m,n)} \\ b^{(p+1,-)}_{(m,n)} \\ \beta^{(p+1,-)}_{(m,n)} \end{bmatrix} - S^{-1}_{(n,p)} \begin{bmatrix} \xi^{(p,1)}_{(m,n)} \\ \xi^{(p,2)}_{(m,n)} \\ \xi^{(p,3)}_{(m,n)} \\ \xi^{(p,4)}_{(m,n)} \end{bmatrix} \tag{4.3.13}$$

Now as there are no sources in the core region we have for the simplest structure with a source in a single shell the relationship

$$\begin{bmatrix} a^{(1,+)}_{(m,n)} \\ 0 \\ b^{(1,+)}_{(m,n)} \\ 0 \end{bmatrix} = Q^{(1)}_n Q^{(2)}_n \begin{bmatrix} a^{(3,-)}_{(m,n)} \\ \alpha^{(3,-)}_{(m,n)} \\ b^{(3,-)}_{(m,n)} \\ \beta^{(3,-)}_{(m,n)} \end{bmatrix} - Q^{(1)}_n S^{-1}_{(n,2)} \begin{bmatrix} \xi^{(2,1)}_{(m,n)} \\ \xi^{(2,2)}_{(m,n)} \\ \xi^{(2,3)}_{(m,n)} \\ \xi^{(2,4)}_{(m,n)} \end{bmatrix} \tag{4.3.14}$$

where the known field representation coefficients $\xi^{(p,1)}_{(m,n)}$, $\xi^{(p,2)}_{(m,n)}$ and $\xi^{(p,4)}_{(m,n)}$ and $\xi^{(p,3)}_{(m,n)}$ are given by equations (4.3.10), (4.3.9), and (4.3.11) respectively. The general relationship is given by

$$
\begin{bmatrix} a^{(1,+)}_{(m,n)} \\ 0 \\ b^{(1,+)}_{(m,n)} \\ 0 \end{bmatrix} = Q^{(1)}_n Q^{(2)}_n \cdots Q^{(N)}_n \begin{bmatrix} a^{(N+1,-)}_{(m,n)} \\ \alpha^{(N+1,-)}_{(m,n)} \\ b^{(N+1,-)}_{(m,n)} \\ \beta^{(N+1,-)}_{(m,n)} \end{bmatrix} -
$$

$$
\sum_{p=1}^{N-1} Q^{(1)}_n Q^{(2)}_n \cdots Q^{(p)}_n S^{-1}_{(n,p+1)} \begin{bmatrix} \xi^{(p,1)}_{(m,n)} \\ \xi^{(p,2)}_{(m,n)} \\ \xi^{(p,3)}_{(m,n)} \\ \xi^{(p,4)}_{(m,n)} \end{bmatrix} \tag{4.3.15}
$$

As before, if the expansion coefficients $a^{(N+1,-)}_{(m,n)}$ and $b^{(N+1,-)}_{(m,n)}$ of the external source are known, then we have a system of 4 equations in 4 unknowns connecting the expansion coefficients in the source free core and the expansion coefficients $\alpha^{(N+1,-)}_{(m,n)}$ and $\beta^{(N+1,-)}_{(m,n)}$ of the radiation scattered by the N layer bianisotropic structure.

5 Energy Balance

5.1 General Considerations

The total power absorbed by a general structure can be determined by a Poynting vector analysis on the surface of the body. The total energy absorbed is the total energy entering the body minus the total energy scattered away from the body.

5.2 Bianisotropy and E H Coupling

In this section we consider the unusual energy balance relationships associated with the interaction of radiation with a bianisotropic material ([13]). The energy balance analysis for an isotropic sphere is carried out in great detail in (Bell [11]). An interchange of dot product and cross product in the triple scalar product implies that the total absorbed

power P_a is given by

$$P_a = (1/2)Re \int\int_{C(R_N)} (\vec{E}_{N+1} \times \vec{H}_{N+1})^* \cdot \vec{n} dA$$

$$= (1/2)Re \int\int_{C(R_N)} \left[(\vec{E}_N \times \vec{H}^*_{N+1}) \cdot \vec{n}\right] dA \tag{5.2.1}$$

where we have used the fact that on the spherical boundary $r = R_N$ we have

$$\vec{n} \times \vec{E}_{N+1} = \vec{n} \times \vec{E}_N \tag{5.2.2}$$

because tangential components of $\vec{E}$ are assumed to be continuous across boundaries separating regions of continuity of tensorial electromagnetic properties. We next make use of the fact that for an impedance boundary condition on the surface of the scattering body that

$$(\vec{H}^*_{N+1} - \vec{H}^*_N) \times \vec{n} = \sigma_s(\vec{E}^*_N - (\vec{E}^*_N \cdot \vec{n})\vec{n}) \tag{5.2.3}$$

where σ_s is the impedance sheet conductivity. From equations (5.2.3) and (5.2.1) we see that

$$P_a = (1/2)Re \int\int_{C(R_N)} (\vec{E}_N \times \vec{H}_N)^* \cdot \vec{n} dA +$$

$$(1/2)Re \int\int_{C(R_N)} \left[\vec{E}_N \cdot (\sigma_s(\vec{E}^*_N - (\vec{E}^*_N \cdot \vec{n})\vec{n}))\right] dA \tag{5.2.4}$$

Using this and the fact that

$$div(\vec{E} \times \vec{H}^*) = \vec{H}^* \cdot curl(\vec{E}) - \vec{E} \cdot curl(\vec{H}^*) \tag{5.2.5}$$

we derive a formula for the internal energy density. For a sweeping beam or a stationary beam interacting with a bianisotropic body or a stationary beam interacting with a moving body (Hebenstreit [29]) there may be unusual couplings of the electromagnetic energy with the structure. For a general one layer structure covered by an impedance sheet the internal energy density is given in terms of the bilinear form

$$b(\vec{E}, \vec{H}) =$$

$$\int_{V_2} \left\{(\vec{E}^*_2 \cdot (i\omega\bar{\bar{\epsilon}} + \bar{\bar{\sigma}})\vec{E}_2) + (\vec{E}_2 \cdot (-i\omega\bar{\bar{\epsilon}}^* + \bar{\bar{\sigma}}^*)\vec{E}^*_2)\right\} dv +$$

$$\int_{V_2} \left\{(\vec{E}^*_2 \cdot (\bar{\bar{\alpha}})\vec{H}_2) + (\vec{E}_2 \cdot (\bar{\bar{\alpha}}^*)\vec{H}^*_2)\right\} dv +$$

$$
\begin{aligned}
&-\int_{V_2}\left\{(\vec{H}_2\cdot(i\omega\overline{\overline{\mu}}^*\vec{H}_2^*)+(\vec{H}_2^*\cdot(-i\omega\overline{\overline{\mu}}\vec{H}_2))\right\}dv+\\
&\int_{V_2}\left\{(\vec{H}_2\cdot\overline{\overline{\beta}}^*\vec{E}_2^*)+(\vec{H}_2^*\cdot\overline{\overline{\beta}}\vec{E}_2)\right\}dv+\\
&\int_{S_2}(\sigma_s^*+\sigma_s)\left\{(\vec{E}_2\cdot\vec{E}_2^*)-(\vec{E}_2\cdot\vec{n})(\vec{E}_2^*\cdot\vec{n})\right\}da
\end{aligned}
\tag{5.2.6}
$$

where S_2 is the bounding surface and V_2 is the interior volume. This can be used as a source term for the heat equation and can be used to predict the response of the structure to a sweeping beam or the response of a moving structure to a stationary beam (Ferencz [25], Gamo [26], Hebenstreit [29], and Shiozawa, [44]). Energy balance computations were carried out in (Bell, Cohoon, and Penn [10], [11]) for isotropic structures and in (Cohoon [15]) for anisotropic structures. These energy balance computations involve comparing the total energy entering the structure minus the total energy reflected from the structure to the sum of the integrals of the power density distributions in the impedance sheets and in the layers themselves.

5.3 Computer Output

```
Electromagnetic Energy Deposition in a Concentric Layered Sphere.

Frequency = 1.000E+03 MHz.

Field Strength =   1.00 V/M   Number of Regions =  2

 Core Radius = 1.1 cm  Shell Radius = 3.3 cm

  Core Properties

                                 ---------------------

Relative Permittivity (Radial):  ( 50.00, 0.00 )
Relative Permittivity (Angular): ( 50.00, 0.00 )
```

```
Relative Permeability (Radial):  ( 2.00, 1.00 )
Relative Permeability (Angular): ( 2.00, 1.00)
Conductivity (Mho/M) (Radial):   ( .600, .600)
Conductivity (Mho/M) (Angular):  ( .600, .600)
Impedance Sheet Cond. (Mho/M):   ( 0.00E+00, 0.00E+00)
Surface Boundary (cm) =              1.1

 Shell Properties

Relative Permittivity (Radial):  ( 30.00, 0.00 )
Relative Permittivity (Angular): ( 60.00, 0.00 )
Relative Permeability (Radial):  ( 2.00, 1.00 )
Relative Permeability (Angular): ( 5.00, 3.00)
Conductivity (Mho/M) (Radial):   ( .200, .600)
Conductivity (Mho/M) (Angular):  ( .400, .600)
Impedance Sheet Cond. (Mho/M):   ( 0.00E+00, 0.00E+00)
Surface Boundary (cm) =              3.3

    Total Absorbed Power = 9.10716094E-6 Watts
 (by Poynting vector analysis on the surface
 and by volume integration of the power density
 over the interior)

    Average Absorbed Power = 6.04996E-2 Watts/Meter**3
```

The fact that the total absorbed power obtained by a Poynting vector method and the total absorbed power obtained by volume integration of the power density distribution nearly coincide represents a confirmation of the correctness of the coding implementing the solution for an anisotropic sphere. The determination of the total absorbed power by the Poynting vector method is described in (Jones [33]) and in full detail in (Bell, [11]). For the plane wave problem described in Jones ([33]) we can give exact formulas for the

total absorbed power in terms of the total power entering the sphere minus the total power scattered away from the sphere ([33], page 504, equation 126). We let $\alpha_{(n,N+1)}$ and $\beta_{(n,N+1)}$ denote the expansion coefficients of the scattered radiation and by carrying out an energy balance book keeping on the boundary we observe that the total absorbed power is

$$P_a = \frac{\pi \mid E_0 \mid^2}{k_0^2}\sqrt{\frac{\epsilon_0}{\mu_0}}\left[Re\sum_{n=1}^{\infty}(2n+1)(\alpha_{(n,N+1)}+\beta_{(n,N+1)})\right]$$

$$-\frac{\pi \mid E_0 \mid^2}{k_0^2}\sqrt{\frac{\epsilon_0}{\mu_0}}\left[\sum_{n=1}^{\infty}(2n+1)\left(\mid \alpha_{(n,N+1)} \mid^2 + \mid \beta_{(n,N+1)} \mid^2\right)\right] \qquad (5.3.1)$$

This is the referred to as the Poynting vector method in the computer output; the last number is the result of numerically integrating the power density distribution over the interior of the sphere. The difficulty of this numerical integration is evident from the following plot of the internal power density distribution for an anisotropic structure with a radial permittivity that is higher than the tangential permittivity.

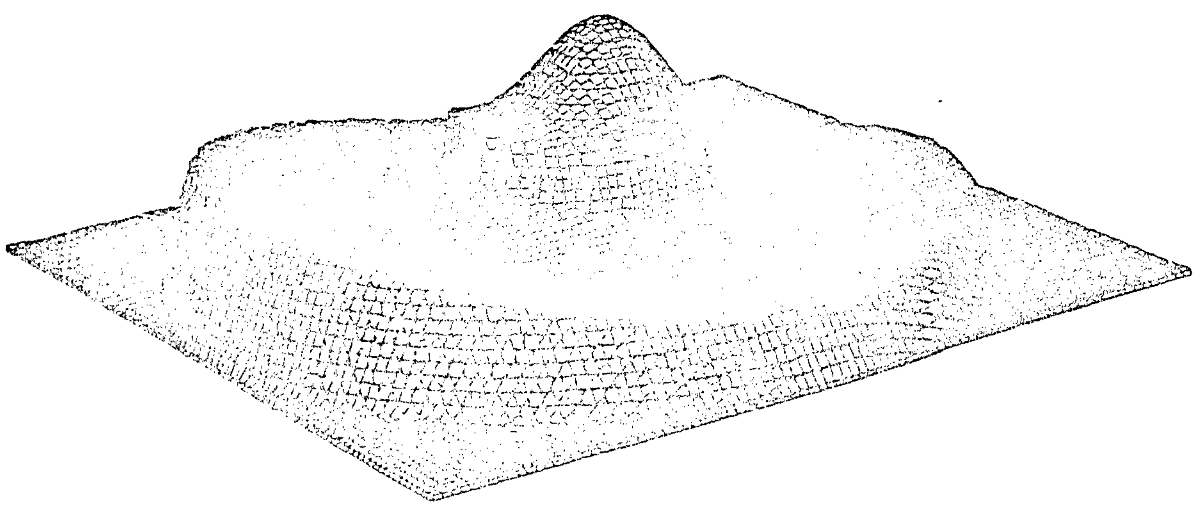


Figure 5.3.1 The electric and magnetic power density distribution on an equatorial slice of a two layer sphere subjected to a single plane wave. The core in this sphere and the one on the following page are identical. The difference is the protective nature of the shell in the following figure.

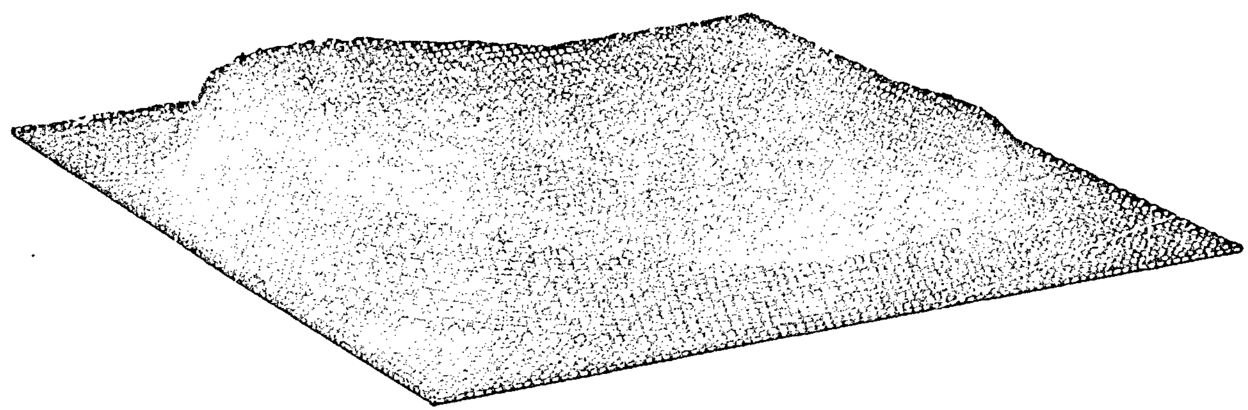


Figure 5.3.2. The electromagnetic power density distribution on an equatorial slice of a two layer sphere subjected to electromagnetic radiation. Here the nature of the shell is such that the electromagnetic energy tends to be shunted away from the core. The ratio of the radius of the core to the radius of the outer shell is 1.1 to 3.3. The radial relative permittivity is 30 and the tangential relative permittivity is 60. These numbers are reversed in Figure 5.3.1.

5.4 Thermal Response to Radiation

The absorption of radiation results in a temperature increase. An energy equation describing this change of state is given by

$$\rho\frac{De}{Dt} = \left(\frac{\partial}{\partial t}\right)Q_{in} + \left(\frac{\partial}{\partial t}\right)Q_{out} +$$
$$(-p\,div(\vec{v})) + div(\overline{\overline{K}}grad(T)) + \Phi, \qquad (5.4.1)$$

where e is equal to c_vT with T denoting the temperature, and c_v denoting the specific heat at constant volume, Φ is the viscous dissipation function (Anderson, Tannehill and Pletcher [1], pages 188-189), $\vec{v}$ is the fluid velocity, ρ is the density, p is the pressure, $\overline{\overline{K}}$ is the tensor thermal conductivity, the term representing the transfer by radiation from one part of the fluid to another is given by (Siegel and Howell [43], page 689)

$$\left(\frac{\partial}{\partial t}\right)Q_{out} = div\left(\frac{16\sigma_e T^3}{3a_R}\cdot grad(T)\right), \qquad (5.4.2)$$

where the internal radiative conductivity is given by

$$k_r = \frac{16\sigma_e T^3}{3a_R}, \qquad (5.4.3)$$

where a_R is the Rosseland mean absorption coefficient (Siegel [43] , p 504 and Rosseland [41]) and where σ_e (Siegel [43], page 25) is the hemispherical total emissive power of a black surface into vacuum having a value of

$$\sigma_e = 5.6696\times 10^{-8} \text{ Watts } / \ (meters^2 \ {}^\circ K\ ), \qquad (5.4.4)$$

and where if $\mathcal{B}(\vec{E},\vec{H})$ represents the absorbed electromagnetic energy per unit volume, whose integral is, (after conversion from cgs units) equal to the $b(\vec{E},\vec{H})$ given by equation (5.2.6) then

$$\left(\frac{\partial}{\partial t}\right)Q_{in} = \mathcal{B}(\vec{E},\vec{H}). \qquad (5.4.5)$$

In general solving equation (5.4.1) requires the simultaneous solution of the Maxwell, continuity, and momentum equations (see Jones [33], p 775). However, for low levels of radiation the energy equation (5.4.1) reduces to a simple heat equation with a source term which can be solved by dovetailing ([12], [14]) it to the solution of the Maxwell equations. The experimental verification of the latter procedure is described in ([12]) and in ([14]).

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Numerical Homotopy and Complex Solutions of $sin(z) = z$

D. K. Cohoon
Department of Mathematics
West Chester University
West Chester, PA 19383

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Since

$$f(z) = sin(z) - z = \frac{exp(iz) - exp(-iz)}{2i} - z \qquad (1)$$

is not of the form $P(z)exp(g(z))$, it follows from Picard's theorem that there must be an infinite number of complex solutions of equation (1). This paper describes a path following or homotopy method for systematically finding these complex zeros as the end point of an orbit of a dynamical system transforming points in the complex plane. Computer plots of these orbits

are presented.

1 Homotopy Methods

We transform the equation,

$$sin(z) = z \tag{1.1}$$

to an equation in another space by using auxiliary functions so that the transformed equation has the form,

$$sin(A(s)z(s)) = (z(s) + B(s)) \tag{1.2}$$

where

$$B(s) = (n \cdot \pi i)(1 - s) \tag{1.3}$$

and

$$z(0) = -n\pi i \tag{1.4}$$

and

$$A(s) = i(1 - s) + s \tag{1.5}$$

so that when s is equal to zero, equation (1.2) has the form,

$$sin(i(-n\pi i)) = sin(n\pi) = (-n\pi i + n\pi i) \tag{1.6}$$

which is true, and when s is equal to 1, then as the trivial equation (1.6) holds at one end of the homotopy path and if equation (1.2) is preserved all the way along the path, and as this equation has the form

$$sin(A(1) \cdot z) = z + B(1) \tag{1.7}$$

at s equal to one, since

$$A(1) = 1 \tag{1.8}$$

and

$$B(1) = 0 \tag{1.9}$$

we see that we obtain a solution of equation (1.1) at the other end of the path.

Thus, the problem is finding a scheme for assuring that the equation (1.2) is preserved all the way along the path. Differentiating both sides of equation (1.2) we see that

$$z'(s) + B'(s) =$$

$$cos(A(s)z(s))\,\{A'(s)\cdot z(s) + A(s)\cdot z'(s)\} \tag{1.10}$$

Collecting terms involving $z'(s)$ we find that

$$\{A(s)cos(A(s)z(s)) - 1\}\,z'(s) =$$

$$B'(s) - z(s)A'(s)cos(A(s)z(s)) \tag{1.11}$$

which leads, after solving equation (1.11), to a coupled system of differential equations in $x(s)$ and $y(s)$ with known values at $s = 0$. Thus,

$$x'(s) = Real\left\{\frac{B'(s) - z(s)A'(s)cos(A(s)z(s))}{A(s)cos(A(s)z(s)) - 1}\right\} \tag{1.12}$$

and

$$y'(s) = Imag\left\{\frac{B'(s) - z(s)A'(s)cos(A(s)z(s))}{A(s)cos(A(s)z(s)) - 1}\right\} \tag{1.13}$$

where

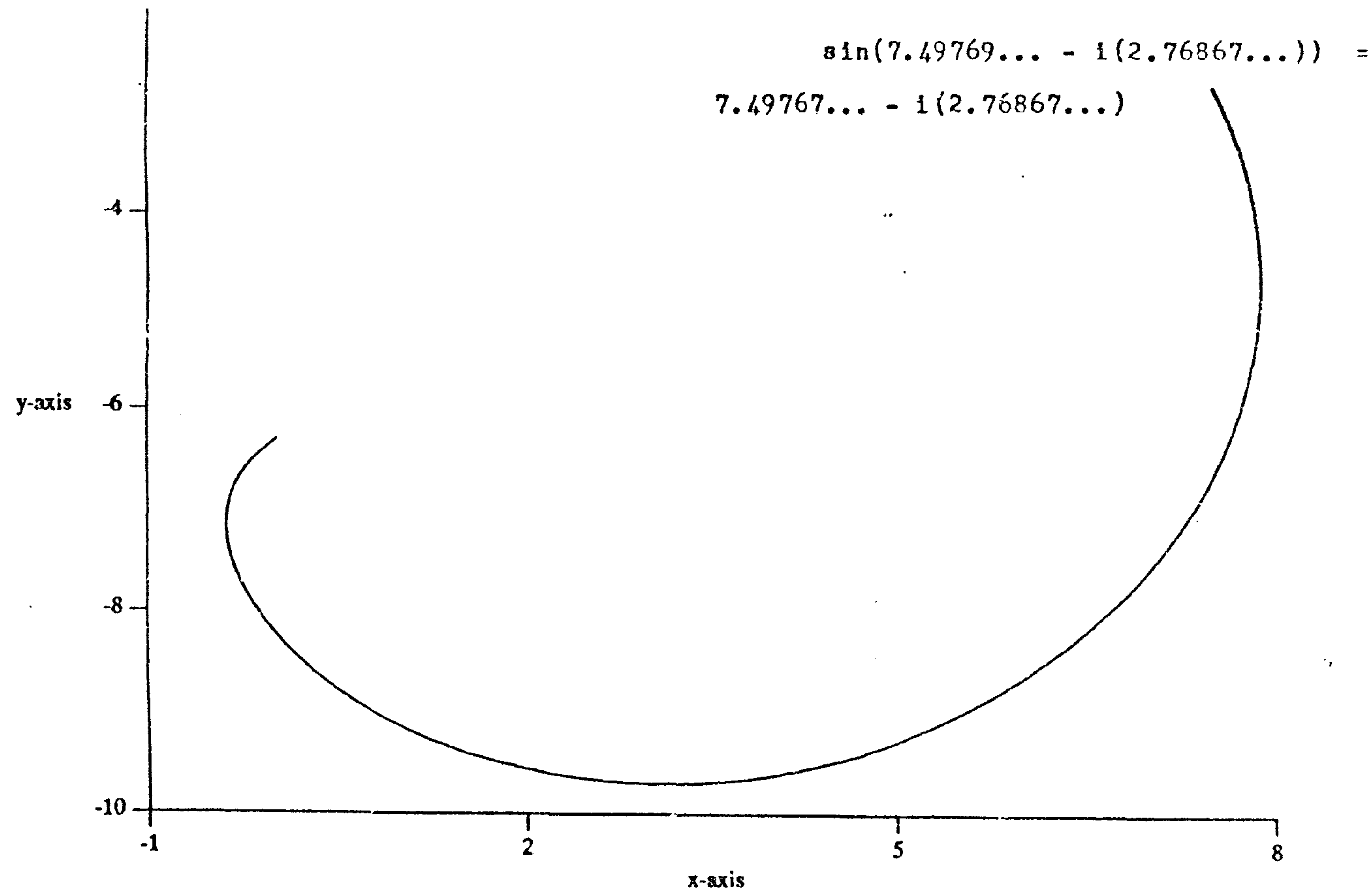
$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -n\pi \end{pmatrix} \tag{1.14}$$

The following graphs give orbits, .

$$s \to (x(s), y(s)) \; , \tag{1.15}$$

representing solutions of the coupled system (1.12) and (1.13) with initial conditions given by equation (1.14). The orbit starting at $(0, -2\pi)$ is shown in the followng figure

Orbit of Homotopy Solution of Sin(z) = z



LINE # TEXT

```
      IMPLICIT REAL*8 (A-H,O-Z)
      COMPLEX*16 CSIN,Z
c     We are systematically finding complex zeros of
c   SIN(Z) = Z using homotopy
      EXTERNAL FYDY,FYDYTST
      DIMENSION IWK(57),ID(3),YARRAY(2),
    1  WK(274),YATST(1)
      COMMON /FYDYCM/NY
      COMMON /PRINT/ICTLOP
      DATA PI/3.14159265358979323D0/
      NSOLN = 2
      ICTLOP = 0
      OPEN(UNIT=11,FILE='ORBIT.DAT',STATUS='UNKNOWN')
      REWIND(11)
c     ICTLOP is an index for counting lines of print
      N = 1
c     N = no of differential equations
      L = 100+21*N
      WRITE(*,*)L,' = L'
      IF(L.GT.274) THEN
        WRITE(*,*)L,' = L is too large'
        STOP
      ENDIF
      T = 0.D0
      TOUT = 1.D0
      NSOLN = 1
      RERR = 1.D-7
      AERR = 1.D-7
      YATST(1) = 1.D0
      IFLAG = -1
      CALL ODE(FYDYTST,N,YATST,T,TOUT,RERR,AERR,IFLAG,WK,
    1 IWK)
      WRITE(*,*)T,' = T'
      WRITE(*,*)TOUT,' = TOUT'
      WRITE(*,*)YATST(1),' = YATST(1)'
      WRITE(*,*)IFLAG,' = IFLAG'
      WRITE(*,*)NSOLN,' = No of solutions'
      N = 2
      L = 100+21*N
      WRITE(*,*)L,' = L'
      IF(L.GT.274) THEN
        WRITE(*,*)L,' = L is too large'
        STOP
      ENDIF
      NSOLN = 1
      NY = 1
      WRITE(*,*)' Input a value of NY'
      CALL GETINT(NY,-4,4)
      DO 3 I = 1,NSOLN
      ID(1) = NY*I
      IFLAG = 1
      YARRAY(1) = 0.D0
      YARRAY(2) = -PI*FLOAT(NY*I)
      WRITE(*,*) ' We are about to call ODE'
      T = 0.D0
      TOUT = 1.D0
      CALL ODE(FYDY,N,YARRAY,
    1 T,TOUT,RERR,AERR,IFLAG,WK,IWK)
      WRITE(*,*)IFLAG,' = IFLAG'
      WRITE(*,*)I,' = solution no completed'
      WRITE(*,*)YARRAY(1),' = YARRAY(1)'
      WRITE(*,*)YARRAY(2),' = YARRAY(2)'
c234567890
      Y1 = YARRAY(1)
      Y2 = YARRAY(2)
      Z = CMPLX(Y1,Y2)
      CSIN = SIN(Z)
      WRITE(*,*)CSIN,' = SIN(Y(1)+iY(2))'
      WRITE(*,*)ICTLOP,' = no of lines of print'
      WRITE(11,*)Y1,Y2
c     This is the terminal point of the orbit
    3 CONTINUE
      CLOSE(UNIT=11)
      STOP
      END
      SUBROUTINE FYDY(T,YARRAY,DYA)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMPLEX*16 CA,CAP,Z,CSOFAZ,CMINPI,
    1 CONE,CDEN,CNUM,CDERIV,CSINAZ,CNOFF,
    2 CZPB
      COMMON /FYDYCM/NY
      COMMON /PRINT/ICTLOP
      DIMENSION YARRAY(2),DYA(2)
c     CMINPI = complex minus I*NY*PI
      DATA ONE,ONEM,ZERO/1.D0,-1.D0,0.D0/
      PI = 3.14159265358979323D0
      X = YARRAY(1)
      Y = YARRAY(2)
      IF(ICTLOP.LT.2000) THEN
        WRITE(11,*)X,Y
      ENDIF
      Z = CMPLX(X,Y)
      CNOFF = CMPLX(ZERO,(ONE-T)*FLOAT(NY)*PI)
      CZPB = Z+CNOFF
      WRITE(*,1122)CZPB
 1122 FORMAT(1PD15.7,D15.7,' = Z + B in FYDY')
      WRITE(*,1133)X,Y
 1133 FORMAT(1PD15.7,D15.7,' = X',D15.7,' = Y')
c
      CA = CMPLX(T,1.D0-T)
      CSINAZ = SIN(CA*Z)
      WRITE(*,1144)CSINAZ
 1144 FORMAT(1PD15.7,D15.7,' = CSINAZ')
      CAP = CMPLX(ONE,ONEM)
      ZERO = 0.D0
      CONE = CMPLX(ONE,ZERO)
      CMINPI = CMPLX(ZERO,-PI*FLOAT(NY))
c     = B'(t)
      CSOFAZ = COS(CA*Z)
      CNUM = CMINPI - CSOFAZ*CAP*Z
      CDEN = CSOFAZ*CA-CONE
      CDERIV = CNUM/CDEN
      DYA(1) = DREAL(CDERIV)
      DYA(2) = DIMAG(CDERIV)
      ICTLOP = ICTLOP + 1
      RETURN
      END

      SUBROUTINE FYDYTST(T,YARRAY,DYA)
      IMPLICIT REAL*8 (A-H,O-Z)
```

LINE #　　　　TEXT

```
      COMMON /PRINT/ICTLOP
      DIMENSION YARRAY(1),DYA(1)
      DATA ONE,OMIN,ZERO/1.D0,-1.D0,0.D0/
      PI = 3.14159265358979323D0
      Y = YARRAY(1)
      DYA(1) = Y
      WRITE(*,*)T,EXP(T),' = T,EXP(T)'
      WRITE(*,*)Y,' = Y'
      WRITE(*,*)DYA(1),' = DYA(1)'
      ICTLOP = ICTLOP+1
      RETURN
      END

      SUBROUTINE ODE(F,NEQN,Y,T,TOUT,RELERR,ABSERR,IFLAG,WORK,IWORK)
C
C     SANDIA MATHEMATICAL PROGRAM LIBRARY
C     APPLIED MATHEMATICS DIVISION 2642
C     SANDIA LABORATORIES
C     ALBUQUERQUE, NEW MEXICO  87115
C     CONTROL DATA 6600 VERSION 6.1  JANUARY 1976
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C
C     WRITTEN BY L. F. SHAMPINE AND M. K. GORDON
C
C     ABSTRACT
C
C
C     SUBROUTINE  ODE  INTEGRATES A SYSTEM OF  NEQN  FIRST ORDER
C     ORDINARY DIFFERENTIAL EQUATIONS OF THE FORM
C               DY(I)/DT = F(T,Y(1),Y(2),...,Y(NEQN))
C               Y(I) GIVEN AT  T .
C     THE SUBROUTINE INTEGRATES FROM  T  TO  TOUT .  ON RETURN THE
C     PARAMETERS IN THE CALL LIST ARE SET FOR CONTINUING THE INTEGRATION.
C     THE USER HAS ONLY TO DEFINE A NEW VALUE  TOUT  AND CALL  ODE  AGAIN.
C
C     THE DIFFERENTIAL EQUATIONS ARE ACTUALLY SOLVED BY A SUITE OF CODES
C     DE1,  STEP1, AND  INTRP .  ODE  ALLOCATES VIRTUAL STORAGE IN THE
C     ARRAYS  WORK  AND  IWORK  AND CALLS  DE1.  DE1 IS A SUPERVISOR WHICH
C     DIRECTS THE SOLUTION.  IT CALLS ON THE ROUTINES  STEP1 AND  INTRP
C     TO ADVANCE THE INTEGRATION AND TO INTERPOLATE AT OUTPUT POINTS.
C     STEP1 USES A MODIFIED DIVIDED DIFFERENCE FORM OF THE ADAMS PECE
C     FORMULAS AND LOCAL EXTRAPOLATION.  IT ADJUSTS THE ORDER AND STEP
C     SIZE TO CONTROL THE LOCAL ERROR PER UNIT STEP IN A GENERALIZED
C     SENSE.  NORMALLY EACH CALL TO  STEP1 ADVANCES THE SOLUTION ONE STEP
C     IN THE DIRECTION OF  TOUT .  FOR REASONS OF EFFICIENCY  DE1
C     INTEGRATES BEYOND  TOUT  INTERNALLY, THOUGH NEVER BEYOND
C     T+10*(TOUT-T), AND CALLS  INTRP  TO INTERPOLATE THE SOLUTION AT
C     TOUT .  AN OPTION IS PROVIDED TO STOP THE INTEGRATION AT  TOUT  BUT
C     IT SHOULD BE USED ONLY IF IT IS IMPOSSIBLE TO CONTINUE THE
C     INTEGRATION BEYOND  TOUT .
C
C     THIS CODE IS COMPLETELY EXPLAINED AND DOCUMENTED IN THE TEXT,
C     COMPUTER SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS:  THE INITIAL
C     VALUE PROBLEM  BY L. F. SHAMPINE AND M. K. GORDON.
C
C     THE PARAMETERS REPRESENT:
C        F -- SUBROUTINE F(T,Y,YP) TO EVALUATE DERIVATIVES YP(I)=DY(I)/DT
C        NEQN -- NUMBER OF EQUATIONS TO BE INTEGRATED
C        Y(*) -- SOLUTION VECTOR AT  T
C        T -- INDEPENDENT VARIABLE
C        TOUT -- POINT AT WHICH SOLUTION IS DESIRED
C        RELERR,ABSERR -- RELATIVE AND ABSOLUTE ERROR TOLERANCES FOR LOCAL
C             ERROR TEST.  AT EACH STEP THE CODE REQUIRES
C               ABS(LOCAL ERROR) .LE. ABS(Y)*RELERR + ABSERR
C             FOR EACH COMPONENT OF THE LOCAL ERROR AND SOLUTION VECTORS
C        IFLAG -- INDICATES STATUS OF INTEGRATION
C        WORK(*),IWORK(*) -- ARRAYS TO HOLD INFORMATION INTERNAL TO CODE
C             WHICH IS NECESSARY FOR SUBSEQUENT CALLS
C
C     FIRST CALL TO ODE --
C
C     THE USER MUST PROVIDE STORAGE IN HIS CALLING PROGRAM FOR THE ARRAYS
C     IN THE CALL LIST,
C        Y(NEQN), WORK(100+21*NEQN), IWORK(5),
C     DECLARE  F  IN AN EXTERNAL STATEMENT, SUPPLY THE SUBROUTINE
C     F(T,Y,YP) TO EVALUATE
C        DY(I)/DT = YP(I) = F(T,Y(1),Y(2),...,Y(NEQN))
C     AND INITIALIZE THE PARAMETERS:
C        NEQN -- NUMBER OF EQUATIONS TO BE INTEGRATED
C        Y(*) -- VECTOR OF INITIAL CONDITIONS
C        T -- STARTING POINT OF INTEGRATION
C        TOUT -- POINT AT WHICH SOLUTION IS DESIRED
C        RELERR,ABSERR -- RELATIVE AND ABSOLUTE LOCAL ERROR TOLERANCES
C        IFLAG -- +1,-1.  INDICATOR TO INITIALIZE THE CODE.  NORMAL INPUT
C             IS +1.  THE USER SHOULD SET IFLAG=-1 ONLY IF IT IS
C             IMPOSSIBLE TO CONTINUE THE INTEGRATION BEYOND  TOUT .
C     ALL PARAMETERS EXCEPT  F ,  NEQN  AND  TOUT  MAY BE ALTERED BY THE
C     CODE ON OUTPUT SO MUST BE VARIABLES IN THE CALLING PROGRAM.
C
C     OUTPUT FROM  ODE  --
C
C        NEQN -- UNCHANGED
C        Y(*) -- SOLUTION AT  T
C        T -- LAST POINT REACHED IN INTEGRATION.  NORMAL RETURN HAS
C             T = TOUT .
C        TOUT -- UNCHANGED
C        RELERR,ABSERR -- NORMAL RETURN HAS TOLERANCES UNCHANGED.  IFLAG=3
C             SIGNALS TOLERANCES INCREASED
C        IFLAG = 2 -- NORMAL RETURN.  INTEGRATION REACHED  TOUT
C              = 3 -- INTEGRATION DID NOT REACH  TOUT  BECAUSE ERROR
C                     TOLERANCES TOO SMALL.  RELERR ,  ABSERR  INCREASED
C                     APPROPRIATELY FOR CONTINUING
C              = 4 -- INTEGRATION DID NOT REACH  TOUT  BECAUSE MORE THAN
```

```
LINE #                                TEXT
C                    RELATIVE ERROR REQUESTED
C           = 5 -- INTEGRATION DID NOT REACH  TOUT  BECAUSE EQUATIONS
C                    APPEAR TO BE STIFF
C           = 6 -- INTEGRATION DID NOT REACH  TOUT  BECAUSE SOLUTION
C                    VANISHED MAKING  PURE RELATIVE ERROR IMPOSSIBLE.
C                    MUST USE NON-ZERO  ABSERR  TO CONTINUE.
C           = 7 -- INVALID INPUT PARAMETERS (FATAL ERROR)
C          THE VALUE OF  IFLAG  IS RETURNED NEGATIVE WHEN THE INPUT
C          VALUE IS NEGATIVE AND THE INTEGRATION DOES NOT REACH  TOUT ,
C          I.E., -3, -4, -5, -6.
C       WORK(*),IWORK(*) -- INFORMATION GENERALLY OF NO INTEREST TO THE
C            USER BUT NECESSARY FOR SUBSEQUENT CALLS.
C
C   SUBSEQUENT CALLS TO  ODE --
C
C   SUBROUTINE  ODE  RETURNS WITH ALL INFORMATION NEEDED TO CONTINUE
C   THE INTEGRATION.  IF THE INTEGRATION REACHED  TOUT , THE USER NEED
C   ONLY DEFINE A NEW  TOUT  AND CALL AGAIN.  IF THE INTEGRATION DID NOT
C   REACH  TOUT  AND THE USER WANTS TO CONTINUE, HE JUST CALLS AGAIN.
C   IN THE CASE  IFLAG=6 , THE USER MUST ALSO ALTER THE ERROR CRITERION.
C   THE OUTPUT VALUE OF  IFLAG  IS THE APPROPRIATE INPUT VALUE FOR
C   SUBSEQUENT CALLS.  THE ONLY SITUATION IN WHICH IT SHOULD BE ALTERED
C   IS TO STOP THE INTEGRATION INTERNALLY AT THE NEW  TOUT , I.E.,
C   CHANGE OUTPUT  IFLAG=2  TO INPUT  IFLAG=-2 .  ERROR TOLERANCES MAY
C   BE CHANGED BY THE USER BEFORE CONTINUING.  ALL OTHER PARAMETERS MUST
C   REMAIN UNCHANGED.
C
      IMPLICIT REAL*8(A-H,O-Z)
      LOGICAL START,PHASE1,NORND
      DIMENSION Y(NEQN),WORK(1),IWORK(5)
      EXTERNAL F
      DATA IALPHA,IBETA,ISIG,IV,IW,IG,IPHASE,IPSI,IX,IH,IHOLD,ISTART,
     1  ITOLD,IDELSN/1,13,25,38,50,62,75,76,88,89,90,91,92,93/
      WRITE(*,*)NEQN,' = NEQN in ODE.f'
      IYY = 100
      IWT = IYY + NEQN
      IP = IWT + NEQN
      IYP = IP + NEQN
      IYPOUT = IYP + NEQN
      IPHI = IYPOUT + NEQN
      WRITE(*,1122)IFLAG
 1122 FORMAT(I5,' = IFLAG in ODE.f')
      IF(IABS(IFLAG) .LT. 2  .OR.  IABS(IFLAG) .GT. 6) GO TO 1
      WRITE(*,1123)ISTART,WORK(ISTART)
 1123 FORMAT(I5,' = ISTART',D15.7,' = WORK(ISTART)')
      START = WORK(ISTART) .GT. 0.0
      WRITE(*,1124)IPHASE,WORK(IPHASE)
 1124 FORMAT(I5,' = IPHASE',D15.7,' = WORK(IPHASE)')
      PHASE1 = WORK(IPHASE) .GT. 0.0
      NORND = IWORK(2) .NE. -1
      WRITE(*,*)' We are about to CALL DE1 in ODE.f'
 1    CALL DE1(F,NEQN,Y,T,TOUT,RELERR,ABSERR,IFLAG,WORK(IYY),
     1  WORK(IWT),WORK(IP),WORK(IYP),WORK(IYPOUT),WORK(IPHI),
     2  WORK(IALPHA),WORK(IBETA),WORK(ISIG),WORK(IV),WORK(IW),WORK(IG),
     3  PHASE1,WORK(IPSI),WORK(IX),WORK(IH),WORK(IHOLD),START,
     4  WORK(ITOLD),WORK(IDELSN),IWORK(1),NORND,IWORK(3),IWORK(4),
     5  IWORK(5))
      WRITE(*,*)' We just exited DE1 in ODE.f'
      WORK(ISTART) = -1.0
      IF(START) WORK(ISTART) = 1.0
      WORK(IPHASE) = -1.0
      IF(PHASE1) WORK(IPHASE) = 1.0
      IWORK(2) = -1
      IF(NORND) IWORK(2) = 1
      RETURN
      END
      SUBROUTINE INTRP(X,Y,XOUT,YOUT,YPOUT,NEQN,KOLD,PHI,PSI)
C
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C   APPLIED MATHEMATICS DIVISION 2642
C   SANDIA LABORATORIES
C   ALBUQUERQUE, NEW MEXICO  87115
C   CONTROL DATA 6600 VERSION 4.1  JANUARY 1976
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C   * THE PRIMARY DOCUMENT FOR THE LIBRARY OF WHICH THIS ROUTINE IS
C   * IS A PART IS SAND75-0545, PRINTED FEB 1976.
C   * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C
C   WRITTEN BY L. F. SHAMPINE AND M. K. GORDON
C
C   ABSTRACT
C
C
C   THE METHODS IN SUBROUTINE  STEP  APPROXIMATE THE SOLUTION NEAR  X
C   BY A POLYNOMIAL.  SUBROUTINE  INTRP  APPROXIMATES THE SOLUTION AT
C   XOUT  BY EVALUATING THE POLYNOMIAL THERE.  INFORMATION DEFINING THIS
C   POLYNOMIAL IS PASSED FROM  STEP  SO  INTRP  CANNOT BE USED ALONE.
C
C   THIS CODE IS COMPLETELY EXPLAINED AND DOCUMENTED IN THE TEXT,
C   COMPUTER SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS:  THE INITIAL
C   VALUE PROBLEM  BY L. F. SHAMPINE AND M. K. GORDON.
C
C   INPUT TO INTRP --
C
C   THE USER PROVIDES STORAGE IN THE CALLING PROGRAM FOR THE ARRAYS IN
C   THE CALL LIST
C      DIMENSION Y(NEQN),YOUT(NEQN),YPOUT(NEQN),PHI(NEQN,16),PSI(12)
C   AND DEFINES
C      XOUT -- POINT AT WHICH SOLUTION IS DESIRED.
C   THE REMAINING PARAMETERS ARE DEFINED IN  STEP  AND PASSED TO  INTRP
C   FROM THAT SUBROUTINE
C
C   OUTPUT FROM  INTRP --
C
C      YOUT(*) -- SOLUTION AT  XOUT
C      YPOUT(*) -- DERIVATIVE OF SOLUTION AT  XOUT
```

```
C     THE REMAINING PARAMETERS ARE RETURNED SO THAT FURTHER INPUT
C     VALUES   INTEGRATION WITH  STEP  MAY BE CONTINUED.
C
        IMPLICIT REAL*8 (A-H,O-Z)
        DIMENSION Y(NEQN),YOUT(NEQN),YPOUT(NEQN),PHI(NEQN,16),PSI(12)
        DIMENSION G(13),W(13),RHO(13)
        DATA G(1)/1.0/,RHO(1)/1.0/
C
        HI = XOUT - X
        KI = KOLD + 1
        KIP1 = KI + 1
C
C     INITIALIZE W(*) FOR COMPUTING G(*)
C
        DO 5 I = 1,KI
          TEMP1 = I
  5       W(I) = 1.0/TEMP1
        TERM = 0.0
C
C     COMPUTE G(*)
C
        DO 15 J = 2,KI
          JM1 = J - 1
          PSIJM1 = PSI(JM1)
          GAMMA = (HI + TERM)/PSIJM1
          ETA = HI/PSIJM1
          LIMIT1 = KIP1 - J
          DO 10 I = 1,LIMIT1
 10         W(I) = GAMMA*W(I) - ETA*W(I+1)
          G(J) = W(1)
          RHO(J) = GAMMA*RHO(JM1)
 15       TERM = PSIJM1
C
C     INTERPOLATE
C
        DO 20 L = 1,NEQN
          YPOUT(L) = 0.0
 20       YOUT(L) = 0.0
        DO 30 J = 1,KI
          I = KIP1 - J
          TEMP2 = G(I)
          TEMP3 = RHO(I)
          DO 25 L = 1,NEQN
            YOUT(L) = YOUT(L) + TEMP2*PHI(L,I)
 25         YPOUT(L) = YPOUT(L) + TEMP3*PHI(L,I)
 30       CONTINUE
        DO 35 L = 1,NEQN
 35       YOUT(L) = Y(L) + HI*YOUT(L)
        RETURN
        END
        SUBROUTINE DE1(F,NEQN,Y,T,TOUT,RELERR,ABSERR,IFLAG,
       1  YY,WT,P,YP,YPOUT,PHI,ALPHA,BETA,SIG,V,W,G,PHASE1,PSI,X,H,HOLD,
       2  START,TOLD,DELSGN,NS,NORND,K,KOLD,ISNOLD)
C
C     DE1  MERELY ALLOCATES STORAGE FOR  DE1 TO RELIEVE THE USER OF THE
C     INCONVENIENCE OF A LONG CALL LIST.  CONSEQUENTLY  DE1 IS USED AS
C     DESCRIBED IN THE COMMENTS FOR  ODE .
C
C     THIS CODE IS COMPLETELY EXPLAINED AND DOCUMENTED IN THE TEXT,
C     COMPUTER SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS:  THE INITIAL
C     VALUE PROBLEM  BY L. F. SHAMPINE AND M. K. GORDON.
C
        IMPLICIT REAL*8 (A-H,O-Z)
        LOGICAL STIFF,CRASH,START,PHASE1,NORND
        DIMENSION Y(NEQN),YY(NEQN),WT(NEQN),PHI(NEQN,16),P(NEQN),YP(NEQN),
       1  YPOUT(NEQN),PSI(12),ALPHA(12),BETA(12),SIG(13),V(12),W(12),G(13)
        EXTERNAL F
C
C     THE CONSTANT  MAXNUM  IS THE MAXIMUM NUMBER OF STEPS ALLOWED IN ONE
C     CALL TO  DE1.  THE USER MAY CHANGE THIS LIMIT BY ALTERING THE
C     FOLLOWING STATEMENT
        DATA MAXNUM/500/
C
C     ****** U IS A MACHINE DEPENDENT PARAMETER. IT IS THE SMALLEST
C            POSITIVE NUMBER FOR WHICH 1.0 + U .GT. 1.0.
C
C       U = EPMPAR(1)
        U = DPMPAR(1)
        FOURU = 4.0*U
C
C            ***            ***                ***
C     TEST FOR IMPROPER PARAMETERS
C
        IF(NEQN .LT. 1) GO TO 10
        IF(T .EQ. TOUT) GO TO 10
        IF(RELERR .LT. 0.0  .OR.  ABSERR .LT. 0.0) GO TO 10
C       EPS = AMAX1(RELERR,ABSERR)
        EPS = DMAX1(RELERR,ABSERR)
        IF(EPS .LE. 0.0) GO TO 10
        IF(IFLAG .EQ. 0) GO TO 10
        ISN = ISIGN(1,IFLAG)
        IFLAG = IABS(IFLAG)
        IF(IFLAG .EQ. 1) GO TO 20
        IF(T .NE. TOLD) GO TO 10
        IF(IFLAG .GE. 2  .AND.  IFLAG .LE. 5) GO TO 20
        IF(IFLAG .EQ. 6  .AND.  ABSERR .GT. 0.0) GO TO 20
 10     IFLAG = 7
        RETURN
C
C     ON EACH CALL SET INTERVAL OF INTEGRATION AND COUNTER FOR NUMBER OF
C     STEPS.  ADJUST INPUT ERROR TOLERANCES TO DEFINE WEIGHT VECTOR FOR
C     SUBROUTINE  STEP1
C
 20     DEL = TOUT - T
        ABSDEL = ABS(DEL)
        TEND = T + 10.0*DEL
        IF(ISN .LT. 0) TEND = TOUT
        NOSTEP = 0
        KLE4 = 0
        STIFF = .FALSE.
        RELEPS = RELERR/EPS
        ABSEPS = ABSERR/EPS
        IF(IFLAG .EQ. 1) GO TO 30
        IF(ISNOLD .LT. 0) GO TO 30
        IF(DELSGN*DEL .GT. 0.0) GO TO 50
C
C     ON START AND RESTART ALSO SET WORK VARIABLES X AND YY(*), STORE THE
C     DIRECTION OF INTEGRATION AND INITIALIZE THE STEP SIZE
C
 30     START = .TRUE.
```

```
      X = T
      DO 40 L = 1,NEQN
   40   YY(L) = Y(L)
      DELSGN = SIGN(1.0,DEL)
C     H = SIGN(AMAX1(FOURU*ABS(X),ABS(TOUT-X)),TOUT-X)
      H = SIGN(DMAX1(FOURU*ABS(X),ABS(TOUT-X)),TOUT-X)
C
C   IF ALREADY PAST OUTPUT POINT, INTERPOLATE AND RETURN
C
   50 IF(ABS(X-T) .LT. ABSDEL) GO TO 60
      CALL INTRP(X,YY,TOUT,Y,YPOUT,NEQN,KOLD,PHI,PSI)
      IFLAG = 2
      T = TOUT
      TOLD = T
      ISNOLD = ISN
      RETURN
C
C   IF CANNOT GO PAST OUTPUT POINT AND SUFFICIENTLY CLOSE,
C   EXTRAPOLATE AND RETURN
C
   60 IF(ISN .GT. 0  .OR.  ABS(TOUT-X) .GE. FOURU*ABS(X)) GO TO 80
      H = TOUT - X
      CALL F(X,YY,YP)
      DO 70 L = 1,NEQN
   70   Y(L) = YY(L) + H*YP(L)
      IFLAG = 2
      T = TOUT
      TOLD = T
      ISNOLD = ISN
      RETURN
C
C   TEST FOR TOO MANY STEPS
C
   80 IF(NOSTEP .LT. MAXNUM) GO TO 100
      IFLAG = ISN*4
      IF(STIFF) IFLAG = ISN*5
      DO 90 L = 1,NEQN
   90   Y(L) = YY(L)
      T = X
      TOLD = T
      ISNOLD = 1
      RETURN
C
C   LIMIT STEP SIZE, SET WEIGHT VECTOR AND TAKE A STEP
C
C100  H = SIGN(AMIN1(ABS(H),ABS(TEND-X)),H)
  100 H = SIGN(DMIN1(ABS(H),ABS(TEND-X)),H)
      DO 110 L = 1,NEQN
        WT(L) = RELEPS*ABS(YY(L)) + ABSEPS
        IF(WT(L) .LE. 0.0) GO TO 140
  110   CONTINUE
      CALL STEP1(F,NEQN,YY,X,H,EPS,WT,START,
     1  HOLD,K,KOLD,CRASH,PHI,P,YP,PSI,
     2  ALPHA,BETA,SIG,V,W,G,PHASE1,NS,NORND)
C
C   TEST FOR TOLERANCES TOO SMALL
C
      IF(.NOT.CRASH) GO TO 130
      IFLAG = ISN*3
      RELERR = EPS*RELEPS
      ABSERR = EPS*ABSEPS
      DO 120 L = 1,NEQN
  120   Y(L) = YY(L)
      T = X
      TOLD = T
      ISNOLD = 1
      RETURN
C
C   AUGMENT COUNTER ON NUMBER OF STEPS AND TEST FOR STIFFNESS
C
  130 NOSTEP = NOSTEP + 1
      KLE4 = KLE4 + 1
      IF(KOLD .GT. 4) KLE4 = 0
      IF(KLE4 .GE. 50) STIFF = .TRUE.
      GO TO 50
C
C   RELATIVE ERROR CRITERION INAPPROPRIATE
C
  140 IFLAG = ISN*6
      DO 150 L = 1,NEQN
  150   Y(L) = YY(L)
      T = X
      TOLD = T
      ISNOLD = 1
      RETURN
      END
      SUBROUTINE STEP1(F,NEQN,Y,X,H,EPS,WT,START,
     1 HOLD,K,KOLD,CRASH,PHI,P,YP,PSI,
     2 ALPHA,BETA,SIG,V,W,G,PHASE1,NS,NORND)
C
C      SANDIA MATHEMATICAL PROGRAM LIBRARY
C      APPLIED MATHEMATICS DIVISION 2642
C      SANDIA LABORATORIES
C      ALBUQUERQUE, NEW MEXICO  87115
C      CONTROL DATA 6600 VERSION 6.1  JANUARY 19..
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C
C   WRITTEN BY L. F. SHAMPINE AND M. K. GORDON
C
C   ABSTRACT
C
C
C   SUBROUTINE  STEP1  IS NORMALLY USED INDIRECTLY THROUGH SUBROUTINE
```

```
C     CODE .  BECAUSE  ODE  SUFFICES FOR MOST PROBLEMS AND IS MUCH EASIER
C     TO USE, USING IT SHOULD BE CONSIDERED BEFORE USING  STEP1  ALONE.
C
C     SUBROUTINE  STEP1  INTEGRATES A SYSTEM OF  NEQN  FIRST ORDER ORDINARY
C     DIFFERENTIAL EQUATIONS ONE STEP, NORMALLY FROM X TO X+H, USING A
C     MODIFIED DIVIDED DIFFERENCE FORM OF THE ADAMS PECE FORMULAS.  LOCAL
C     EXTRAPOLATION IS USED TO IMPROVE ABSOLUTE STABILITY AND ACCURACY.
C     THE CODE ADJUSTS ITS ORDER AND STEP SIZE TO CONTROL THE LOCAL ERROR
C     PER UNIT STEP IN A GENERALIZED SENSE.  SPECIAL DEVICES ARE INCLUDED
C     TO CONTROL ROUNDOFF ERROR AND TO DETECT WHEN THE USER IS REQUESTING
C     TOO MUCH ACCURACY.
C
C     THIS CODE IS COMPLETELY EXPLAINED AND DOCUMENTED IN THE TEXT,
C     COMPUTER SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS0  THE INITIAL
C     VALUE PROBLEM  BY L. F. SHAMPINE AND M. K. GORDON.
C
C
C     THE PARAMETERS REPRESENT)
C        F -- SUBROUTINE TO EVALUATE DERIVATIVES
C        NEQN -- NUMBER OF EQUATIONS TO BE INTEGRATED
C        Y(*) -- SOLUTION VECTOR AT X
C        X -- INDEPENDENT VARIABLE
C        H -- APPROPRIATE STEP SIZE FOR NEXT STEP.  NORMALLY DETERMINED BY
C             CODE
C        EPS -- LOCAL ERROR TOLERANCE
C        WT(*) -- VECTOR OF WEIGHTS FOR ERROR CRITERION
C        START -- LOGICAL VARIABLE SET .TRUE. FOR FIRST STEP,  .FALSE.
C             OTHERWISE
C        HOLD -- STEP SIZE USED FOR LAST SUCCESSFUL STEP
C        K -- APPROPRIATE ORDER FOR NEXT STEP (DETERMINED BY CODE)
C        KOLD -- ORDER USED FOR LAST SUCCESSFUL STEP
C        CRASH -- LOGICAL VARIABLE SET .TRUE. WHEN NO STEP CAN BE TAKEN,
C             .FALSE. OTHERWISE
C        YP(*) -- DERIVATIVE OF SOLUTION VECTOR AT  X  AFTER SUCCESSFUL
C             STEP
C     THE ARRAYS  PHI, PSI  ARE REQUIRED FOR THE INTERPOLATION SUBROUTINE
C     INTRP .  THE ARRAY  P  IS INTERNAL TO THE CODE.  THE REMAINING NINE
C     VARIABLES AND ARRAYS ARE INCLUDED IN THE CALL LIST ONLY TO ELIMINATE
C     LOCAL RETENTION OF VARIABLES BETWEEN CALLS.
C
C     INPUT TO STEP1
C
C        FIRST CALL --
C
C     THE USER MUST PROVIDE STORAGE IN HIS CALLING PROGRAM FOR ALL ARRAYS
C     IN THE CALL LIST, NAMELY
C
C       DIMENSION Y(NEQN),WT(NEQN),PHI(NEQN,16),P(NEQN),YP(NEQN),PSI(12),
C      1  ALPHA(12),BETA(12),SIG(13),V(12),W(12),G(13)
C
C     THE USER MUST ALSO DECLARE  START ,  CRASH ,  PHASE1  AND  NORND
C     LOGICAL VARIABLES AND  F  AN EXTERNAL SUBROUTINE, SUPPLY THE
C     SUBROUTINE  F(X,Y,YP)  TO EVALUATE
C        DY(I)/DX = YP(I) = F(X,Y(1),Y(2),...,Y(NEQN))
C     AND INITIALIZE ONLY THE FOLLOWING PARAMETERS0
C        NEQN -- NUMBER OF EQUATIONS TO BE INTEGRATED
C        Y(*) -- VECTOR OF INITIAL VALUES OF DEPENDENT VARIABLES
C        X -- INITIAL VALUE OF THE INDEPENDENT VARIABLE
C        H -- NOMINAL STEP SIZE INDICATING DIRECTION OF INTEGRATION
C             AND MAXIMUM SIZE OF STEP.  MUST BE VARIABLE
C        EPS -- LOCAL ERROR TOLERANCE PER STEP.  MUST BE VARIABLE
C        WT(*) -- VECTOR OF NON-ZERO WEIGHTS FOR ERROR CRITERION
C        START -- .TRUE.
C
C     STEP1  REQUIRES THAT THE L2 NORM OF THE VECTOR WITH COMPONENTS
C     LOCAL ERROR(L)/WT(L)  BE LESS THAN  EPS  FOR A SUCCESSFUL STEP.  THE
C     ARRAY  WT  ALLOWS THE USER TO SPECIFY AN ERROR TEST APPROPRIATE
C     FOR HIS PROBLEM.  FOR EXAMPLE,
C        WT(L) = 1.0  SPECIFIES ABSOLUTE ERROR,
C              = ABS(Y(L))  ERROR RELATIVE TO THE MOST RECENT VALUE OF THE
C                   L-TH COMPONENT OF THE SOLUTION,
C              = ABS(YP(L))  ERROR RELATIVE TO THE MOST RECENT VALUE OF
C                   THE L-TH COMPONENT OF THE DERIVATIVE,
C              = AMAX1(WT(L),ABS(Y(L)))  ERROR RELATIVE TO THE LARGEST
C                   MAGNITUDE OF L-TH COMPONENT OBTAINED SO FAR,
C              = ABS(Y(L))*RELERR/EPS + ABSERR/EPS  SPECIFIES A MIXED
C                   RELATIVE-ABSOLUTE TEST WHERE  RELERR  IS RELATIVE
C                   ERROR,  ABSERR  IS ABSOLUTE ERROR AND  EPS =
C                   AMAX1(RELERR,ABSERR) .
C
C        SUBSEQUENT CALLS --
C
C     SUBROUTINE  STEP1  IS DESIGNED SO THAT ALL INFORMATION NEEDED TO
C     CONTINUE THE INTEGRATION, INCLUDING THE STEP SIZE  H  AND THE ORDER
C     K , IS RETURNED WITH EACH STEP.  WITH THE EXCEPTION OF THE STEP
C     SIZE, THE ERROR TOLERANCE, AND THE WEIGHTS, NONE OF THE PARAMETERS
C     SHOULD BE ALTERED.  THE ARRAY  WT  MUST BE UPDATED AFTER EACH STEP
C     TO MAINTAIN RELATIVE ERROR TESTS LIKE THOSE ABOVE.  NORMALLY THE
C     INTEGRATION IS CONTINUED JUST BEYOND THE DESIRED ENDPOINT AND THE
C     SOLUTION INTERPOLATED THERE WITH SUBROUTINE  INTRP .  IF IT IS
C     IMPOSSIBLE TO INTEGRATE BEYOND THE ENDPOINT, THE STEP SIZE MAY BE
C     REDUCED TO HIT THE ENDPOINT SINCE THE CODE WILL NOT TAKE A STEP
C     LARGER THAN THE  H  INPUT.  CHANGING THE DIRECTION OF INTEGRATION,
C     I.E., THE SIGN OF  H , REQUIRES THE USER SET  START = .TRUE. BEFORE
C     CALLING  STEP1  AGAIN.  THIS IS THE ONLY SITUATION IN WHICH  START
C     SHOULD BE ALTERED.
C
C     OUTPUT FROM STEP1
C
C        SUCCESSFUL STEP --
C
C     THE SUBROUTINE RETURNS AFTER EACH SUCCESSFUL STEP WITH  START  AND
C     CRASH  SET .FALSE. .  X  REPRESENTS THE INDEPENDENT VARIABLE
C     ADVANCED ONE STEP OF LENGTH  HOLD  FROM ITS VALUE ON INPUT AND  Y
C     THE SOLUTION VECTOR AT THE NEW VALUE OF  X .  ALL OTHER PARAMETERS
C     REPRESENT INFORMATION CORRESPONDING TO THE NEW  X  NEEDED TO
C     CONTINUE THE INTEGRATION.
C
C        UNSUCCESSFUL STEP --
C
C     WHEN THE ERROR TOLERANCE IS TOO SMALL FOR THE MACHINE PRECISION,
C     THE SUBROUTINE RETURNS WITHOUT TAKING A STEP AND  CRASH = .TRUE. .
C     AN APPROPRIATE STEP SIZE AND ERROR TOLERANCE FOR CONTINUING ARE
C     ESTIMATED AND ALL OTHER INFORMATION IS RESTORED AS UPON INPUT
C     BEFORE RETURNING.  TO CONTINUE WITH THE LARGER TOLERANCE, THE USER
C     JUST CALLS THE CODE AGAIN.  A RESTART IS NEITHER REQUIRED NOR
C     DESIRABLE.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      LOGICAL START,CRASH,PHASE1,NORND
```

LINE #	TEXT

```
      DIMENSION Y(NEQN),WT(NEQN),PHI(NEQN,16),P(NEQN),YP(NEQN),PSI(12),
     1  ALPHA(12),BETA(12),SIG(13),V(12),W(12),G(13)
      DIMENSION TWO(13),GSTR(13)
      EXTERNAL F
C
      DATA TWO(1)/2.0/, TWO(2)/4.0/, TWO(3)/8.0/, TWO(4)/16.0/,
     1     TWO(5)/32.0/, TWO(6)/64.0/, TWO(7)/128.0/, TWO(8)/256.0/,
     2     TWO(9)/512.0/, TWO(10)/1024.0/, TWO(11)/2048.0/,
     3     TWO(12)/4096.0/, TWO(13)/8192.0/
      DATA GSTR(1)/0.500/, GSTR(2)/0.0833/, GSTR(3)/0.0417/,
     1     GSTR(4)/0.0264/, GSTR(5)/0.0188/, GSTR(6)/0.0143/,
     2     GSTR(7)/0.0114/, GSTR(8)/0.00936/, GSTR(9)/0.00789/,
     3     GSTR(10)/0.00679/, GSTR(11)/0.00592/, GSTR(12)/0.00524/,
     4     GSTR(13)/0.00468/
C
C    ****** U IS A MACHINE DEPENDENT PARAMETER. IT IS THE SMALLEST
C           POSITIVE NUMBER FOR WHICH 1.0 + U .GT. 1.0.
C
C     U = SPMPAR(1)
      U = DPMPAR(1)
      TWOU = 2.0*U
      FOURU = 4.0*U
C
C
C       ***     BEGIN BLOCK 0     ***
C   CHECK IF STEP SIZE OR ERROR TOLERANCE IS TOO SMALL FOR MACHINE
C   PRECISION.  IF FIRST STEP, INITIALIZE PHI ARRAY AND ESTIMATE A
C   STARTING STEP SIZE.
C                   ***
C
C   IF STEP SIZE IS TOO SMALL, DETERMINE AN ACCEPTABLE ONE
C
      CRASH = .TRUE.
      IF(ABS(H) .GE. FOURU*ABS(X)) GO TO 5
      H = SIGN(FOURU*ABS(X),H)
      RETURN
 5    P5EPS = 0.5*EPS
C
C   IF ERROR TOLERANCE IS TOO SMALL, INCREASE IT TO AN ACCEPTABLE VALUE
C
      ROUND = 0.0
      DO 10 L = 1,NEQN
 10     ROUND = ROUND + (Y(L)/WT(L))**2
      ROUND = TWOU*SQRT(ROUND)
      IF(P5EPS .GE. ROUND) GO TO 15
      EPS = 2.0*ROUND*(1.0 + FOURU)
      RETURN
 15   CRASH = .FALSE.
      G(1) = 1.0
      G(2) = 0.5
      SIG(1) = 1.0
      IF(.NOT.START) GO TO 99
C
C   INITIALIZE.  COMPUTE APPROPRIATE STEP SIZE FOR FIRST STEP
C
      CALL F(X,Y,YP)
      SUM = 0.0
      DO 20 L = 1,NEQN
        PHI(L,1) = YP(L)
        PHI(L,2) = 0.0
 20     SUM = SUM + (YP(L)/WT(L))**2
      SUM = SQRT(SUM)
      ABSH = ABS(H)
      IF(EPS .LT. 16.0*SUM*H*H) ABSH = 0.25*SQRT(EPS/SUM)
C     H = SIGN(AMAX1(ABSH,FOURU*ABS(X)),H)
      H = SIGN(DMAX1(ABSH,FOURU*ABS(X)),H)
      HOLD = 0.0
      K = 1
      KOLD = 0
      START = .FALSE.
      PHASE1 = .TRUE.
      NORND = .TRUE.
      IF(P5EPS .GT. 100.0*ROUND) GO TO 99
      NORND = .FALSE.
      DO 25 L = 1,NEQN
 25     PHI(L,15) = 0.0
 99   IFAIL = 0
C       ***     END BLOCK 0     ***
C
C       ***     BEGIN BLOCK 1     ***
C   COMPUTE COEFFICIENTS OF FORMULAS FOR THIS STEP.  AVOID COMPUTING
C   THOSE QUANTITIES NOT CHANGED WHEN STEP SIZE IS NOT CHANGED.
C                   ***
C
 100  KP1 = K+1
      KP2 = K+2
      KM1 = K-1
      KM2 = K-2
C
C   NS IS THE NUMBER OF STEPS TAKEN WITH SIZE H, INCLUDING THE CURRENT
C   ONE.  WHEN K.LT.NS, NO COEFFICIENTS CHANGE
C
      IF(H .NE. HOLD) NS = 0
      IF (NS.LE.KOLD) NS = NS+1
      NSP1 = NS+1
      IF (K .LT. NS) GO TO 199
C
C   COMPUTE THOSE COMPONENTS OF ALPHA(*),BETA(*),PSI(*),SIG(*) WHICH
C   ARE CHANGED
C
      BETA(NS) = 1.0
      REALNS = NS
      ALPHA(NS) = 1.0/REALNS
      TEMP1 = H*REALNS
      SIG(NSP1) = 1.0
      IF(K .LT. NSP1) GO TO 110
      DO 105 I = NSP1,K
        IM1 = I-1
        TEMP2 = PSI(IM1)
        PSI(IM1) = TEMP1
        BETA(I) = BETA(IM1)*PSI(IM1)/TEMP2
        TEMP1 = TEMP2 + H
        ALPHA(I) = H/TEMP1
        REALI = I
 105    SIG(I+1) = REALI*ALPHA(I)*SIG(I)
 110  PSI(K) = TEMP1
C
C   COMPUTE COEFFICIENTS G(*)
C
C   INITIALIZE V(*) AND SET W(*)
```

LINE # TEXT

```
C
      IF(NS .GT. 1) GO TO 120
      DO 115 IQ = 1,K
        TEMP3 = IQ*(IQ+1)
        V(IQ) = 1.0/TEMP3
 115    W(IQ) = V(IQ)
      GO TO 140
C
C   IF ORDER WAS RAISED, UPDATE DIAGONAL PART OF V(*)
C
 120  IF(K .LE. KOLD) GO TO 130
      TEMP4 = K*KP1
      V(K) = 1.0/TEMP4
      NSM2 = NS-2
      IF(NSM2 .LT. 1) GO TO 130
      DO 125 J = 1,NSM2
        I = K-J
 125    V(I) = V(I) - ALPHA(J+1)*V(I+1)
C
C   UPDATE V(*) AND SET W(*)
C
 130  LIMIT1 = KP1 - NS
      TEMP5 = ALPHA(NS)
      DO 135 IQ = 1,LIMIT1
        V(IQ) = V(IQ) - TEMP5*V(IQ+1)
 135    W(IQ) = V(IQ)
      G(NSP1) = W(1)
C
C   COMPUTE THE G(*) IN THE WORK VECTOR W(*)
C
 140  NSP2 = NS + 2
      IF(KP1 .LT. NSP2) GO TO 199
      DO 150 I = NSP2,KP1
        LIMIT2 = KP2 - I
        TEMP6 = ALPHA(I-1)
        DO 145 IQ = 1,LIMIT2
 145      W(IQ) = W(IQ) - TEMP6*W(IQ+1)
 150    G(I) = W(1)
 199    CONTINUE
C       ***        END BLOCK 1      ***
C
C       ***        BEGIN BLOCK 2      ***
C   PREDICT A SOLUTION P(*), EVALUATE DERIVATIVES USING PREDICTED
C   SOLUTION, ESTIMATE LOCAL ERROR AT ORDER K AND ERRORS AT ORDERS K,
C   K-1, K-2 AS IF CONSTANT STEP SIZE WERE USED.
C                     ***
C
C   CHANGE PHI TO PHI STAR
C
      IF(K .LT. NSP1) GO TO 215
      DO 210 I = NSP1,K
        TEMP1 = BETA(I)
        DO 205 L = 1,NEQN
 205      PHI(L,I) = TEMP1*PHI(L,I)
 210    CONTINUE
C
C   PREDICT SOLUTION AND DIFFERENCES
C
 215  DO 220 L = 1,NEQN
        PHI(L,KP2) = PHI(L,KP1)
        PHI(L,KP1) = 0.0
 220    P(L) = 0.0
      DO 230 J = 1,K
        I = KP1 - J
        IP1 = I+1
        TEMP2 = G(I)
        DO 225 L = 1,NEQN
          P(L) = P(L) + TEMP2*PHI(L,I)
 225      PHI(L,I) = PHI(L,I) + PHI(L,IP1)
 230    CONTINUE
      IF(NORND) GO TO 240
      DO 235 L = 1,NEQN
        TAU = H*P(L) - PHI(L,15)
        P(L) = Y(L) + TAU
 235    PHI(L,16) = (P(L) - Y(L)) - TAU
      GO TO 250
 240  DO 245 L = 1,NEQN
 245    P(L) = Y(L) + H*P(L)
 250  XOLD = X
      X = X + H
      ABSH = ABS(H)
      CALL F(X,P,YP)
C
C   ESTIMATE ERRORS AT ORDERS K,K-1,K-2
C
      ERKM2 = 0.0
      ERKM1 = 0.0
      ERK = 0.0
      DO 265 L = 1,NEQN
        TEMP3 = 1.0/WT(L)
        TEMP4 = YP(L) - PHI(L,1)
        IF(KM2)265,260,255
 255    ERKM2 = ERKM2 + ((PHI(L,KM1)+TEMP4)*TEMP3)**2
 260    ERKM1 = ERKM1 + ((PHI(L,K)+TEMP4)*TEMP3)**2
 265    ERK = ERK + (TEMP4*TEMP3)**2
      IF(KM2)280,275,270
 270  ERKM2 = ABSH*SIG(KM1)*GSTR(KM2)*SQRT(ERKM2)
 275  ERKM1 = ABSH*SIG(K)*GSTR(KM1)*SQRT(ERKM1)
 280  TEMP5 = ABSH*SQRT(ERK)
      ERR = TEMP5*(G(K)-G(KP1))
      ERK = TEMP5*SIG(KP1)*GSTR(K)
      KNEW = K
C
C   TEST IF ORDER SHOULD BE LOWERED
C
      IF(KM2)299,290,285
c285  IF(AMAX1(ERKM1,ERKM2) .LE. ERK) KNEW = KM1
 285  IF(DMAX1(ERKM1,ERKM2) .LE. ERK) KNEW = KM1
      GO TO 299
 290  IF(ERKM1 .LE. 0.5*ERK) KNEW = KM1
C
C   TEST IF STEP SUCCESSFUL
C
 299  IF(ERR .LE. EPS) GO TO 400
C       ***        END BLOCK 2      ***
C
C       ***        BEGIN BLOCK 3      ***
C   THE STEP IS UNSUCCESSFUL.  RESTORE  X, PHI(*,*), PSI(*) .
C   IF THIRD CONSECUTIVE FAILURE, SET ORDER TO ONE.  IF STEP FAILS MORE
C   THAN THREE TIMES, CONSIDER AN OPTIMAL STEP SIZE.  DOUBLE ERROR
```

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C     --      END BLOCK 4     ***
      END
      SUBROUTINE GETINT(I,LSMALL,LBIG)
      IMPLICIT REAL*8 (A-H, O-Z)
    5 READ(*,*,ERR=20) I
      IF (I.LT.LSMALL .OR. I.GT.LBIG) THEN
        WRITE(*,10)LSMALL,LBIG
   10   FORMAT(' **** You must specify an integer value between'/
     1         ' ',I6,' and ',I6,' ****'/' Please try again ?')
        GOTO 5
      ELSE
        RETURN
      ENDIF
   20 WRITE(*,30)
   30 FORMAT(' **** You must enter an integer here ****'/
     1        ' Please try again ? ')
      GOTO 5
      END
      DOUBLE PRECISION FUNCTION DPMPAR(I)
      INTEGER I
C----------------------------------------------------------------------
C
C     DPMPAR PROVIDES THE DOUBLE PRECISION MACHINE PARAMETERS FOR
C     THE COMPUTER BEING USED. IT IS ASSUMED THAT THE ARGUMENT
C     I IS AN INTEGER HAVING ONE OF THE VALUES 1, 2, OR 3. IF THE
C     DOUBLE PRECISION ARITHMETIC BEING USED HAS T BASE B DIGITS AND
C     ITS SMALLEST AND LARGEST EXPONENTS ARE EMIN AND EMAX. THEN
C
C        DPMPAR(1) = B**(1 - T), THE MACHINE PRECISION,
C
C        DPMPAR(2) = B**(EMIN - 1), THE SMALLEST MAGNITUDE,
C
C        DPMPAR(3) = B**EMAX*(1 - B**(-T)), THE LARGEST MAGNITUDE.
C
C     TO DEFINE THIS FUNCTION FOR THE COMPUTER BEING USED, ACTIVATE
C     THE DATA STATEMENTS FOR THE COMPUTER BY REMOVING THE C FROM
C     COLUMN 1. (ALL OTHER DATA STATEMENTS IN DPMPAR SHOULD HAVE C
C     IN COLUMN 1.) IF DATA STATEMENTS ARE NOT GIVEN FOR THE COMPUTER
C     BEING USED, THEN THE SUBROUTINE MACH MAY BE USED TO COMPUTE THE
C     VALUES FOR DPMPAR.
C
C----------------------------------------------------------------------
C
C     DPMPAR IS AN ADAPTATION OF THE FUNCTION D1MACH, WRITTEN BY P.A.
C     FOX, A.D. HALL, AND N.L. SCHRYER (BELL LABORATORIES). DPMPAR
C     WAS DESIGNED BY B.S. GARBOW, K.E. HILLSTROM, AND J.J. MORE
C     (ARGONNE NATIONAL LABORATORY). THE MAJORITY OF PARAMETER VALUES
C     ARE FROM BELL LABORATORIES.
C
C----------------------------------------------------------------------
      INTEGER MCHEPS(4)
      INTEGER MINMAG(4)
      INTEGER MAXMAG(4)
      DOUBLE PRECISION DMACH(3)
      EQUIVALENCE (DMACH(1),MCHEPS(1))
      EQUIVALENCE (DMACH(2),MINMAG(1))
      EQUIVALENCE (DMACH(3),MAXMAG(1))
C
C     MACHINE CONSTANTS FOR THE BURROUGHS 1700 SYSTEM.
C
C     DATA MCHEPS(1) / ZCC6800000 /
C     DATA MCHEPS(2) / Z000000000 /
C
C     DATA MINMAG(1) / ZC00800000 /
C     DATA MINMAG(2) / Z000000000 /
C
C     DATA MAXMAG(1) / ZDFFFFFFFF /
C     DATA MAXMAG(2) / ZFFFFFFFFF /
C
C     MACHINE CONSTANTS FOR THE BURROUGHS 5700 SYSTEM.
C
C     DATA MCHEPS(1) / O1451000000000000 /
C     DATA MCHEPS(2) / O0000000000000000 /
C
C     DATA MINMAG(1) / O1771000000000000 /
C     DATA MINMAG(2) / O0000000000000000 /
C
C     DATA MAXMAG(1) / O0777777777777777 /
C     DATA MAXMAG(2) / O0007777777777777 /
C
C     MACHINE CONSTANTS FOR THE BURROUGHS 6700/7700 SYSTEMS.
C
C     DATA MCHEPS(1) / O1451000000000000 /
C     DATA MCHEPS(2) / O0000000000000000 /
C
C     DATA MINMAG(1) / O1771000000000000 /
C     DATA MINMAG(2) / O7770000000000000 /
C
C     DATA MAXMAG(1) / O0777777777777777 /
C     DATA MAXMAG(2) / O7777777777777777 /
C
C     MACHINE CONSTANTS FOR THE CDC 6000/7000 SERIES.
C     (OCTAL FORMAT FOR FORTRAN 4 COMPILERS)
C
C     DATA MCHEPS(1) / 15614000000000000000B /
C     DATA MCHEPS(2) / 15010000000000000000B /
C
C     DATA MINMAG(1) / 00014000000000000000B /
C     DATA MINMAG(2) / 00000000000000000000B /
C
C     DATA MAXMAG(1) / 37767777777777777777B /
C     DATA MAXMAG(2) / 37167777777777777777B /
C
C     MACHINE CONSTANTS FOR THE CDC 6000/7000 SERIES.
C     (INTEGER FORMAT FOR FORTRAN 4 AND 5 COMPILERS)
C
      DATA MCHEPS(1) / 2481201919704433184 /
      DATA MCHEPS(2) / 2344888555999744448 /
C
      DATA MINMAG(1) /     422212465745984 /
      DATA MINMAG(2) /                   0 /
C
      DATA MAXMAG(1) / 376179377336712831 /
      DATA MAXMAG(2) / 562848478444601363 /
C
C     MACHINE CONSTANTS FOR THE CRAY-1.
C
C     DATA MCHEPS(1) / 0376424000000000000000B /
C     DATA MCHEPS(2) / 0000000000000000000000B /
C
```

```
C     DATA MINMAG(1) / [illegible]B /
C     DATA MINMAG(2) / [illegible]B /
C
C     DATA MAXMAG(1) / 0577777777777777777777B /
C     DATA MAXMAG(2) / 0000007777777777777776B /
C
C     MACHINE CONSTANTS FOR THE DATA GENERAL ECLIPSE S/200.
C
C     NOTE - IT MAY BE APPROPRIATE TO INCLUDE THE FOLLOWING CARD -
C     STATIC DMACH(3)
C
C     DATA MINMAG/20K,3*0/,MAXMAG/77777K,3*177777K/
C     DATA MCHEPS/32020K,3*0/
C
C     MACHINE CONSTANTS FOR THE HARRIS 220.
C
C     DATA MCHEPS(1),MCHEPS(2) / '20000000, '00000334 /
C     DATA MINMAG(1),MINMAG(2) / '20000000, '00000201 /
C     DATA MAXMAG(1),MAXMAG(2) / '37777777, '37777577 /
C
C     MACHINE CONSTANTS FOR THE HONEYWELL 600/6000 SERIES.
C
C     DATA MCHEPS(1),MCHEPS(2) / 0606400000000, 0000000000000 /
C     DATA MINMAG(1),MINMAG(2) / 0402400000000, 0000000000000 /
C     DATA MAXMAG(1),MAXMAG(2) / 0376777777777, 0777777777777 /
C
C     MACHINE CONSTANTS FOR THE HP 2100
C     THREE WORD DOUBLE PRECISION OPTION WITH FTN4
C
C     DATA MCHEPS(1), MCHEPS(2), MCHEPS(3) / 40000B,       0,    276B /
C     DATA MINMAG(1), MINMAG(2), MINMAG(3) / 40000B,       0,      1 /
C     DATA MAXMAG(1), MAXMAG(2), MAXMAG(3) / 77777B, 177777B, 177776B /
C
C     MACHINE CONSTANTS FOR THE HP 2100
C     FOUR WORD DOUBLE PRECISION OPTION WITH FTN4
C
C     DATA MCHEPS(1), MCHEPS(2) /  40000B,       0 /
C     DATA MCHEPS(3), MCHEPS(4) /       0,    227B /
C     DATA MINMAG(1), MINMAG(2) /  40000B,       0 /
C     DATA MINMAG(3), MINMAG(4) /       0,       1 /
C     DATA MAXMAG(1), MAXMAG(2) /  77777B, 177777B /
C     DATA MAXMAG(3), MAXMAG(4) / 177777B, 177776B /
C
C     MACHINE CONSTANTS FOR THE HP 9000
C
C     DATA DMACH(1) / .22204460492503131D-15  /
C     DATA DMACH(2) / .22250738585072014D-307 /
C     DATA DMACH(3) / .17976931348623157D+309 /
C
C     MACHINE CONSTANTS FOR THE IBM 360/370 SERIES,
C     THE AMDAHL 470/V6, THE ICL 2900, THE ITEL AS/6,
C     THE XEROX SIGMA 5/7/9 AND THE SEL SYSTEMS 85/86.
C
C     DATA MCHEPS(1),MCHEPS(2) / Z34100000, Z00000000 /
C     DATA MINMAG(1),MINMAG(2) / Z00100000, Z00000000 /
C     DATA MAXMAG(1),MAXMAG(2) / Z7FFFFFFF, ZFFFFFFFF /
C
C     MACHINE CONSTANTS FOR THE IBM PC - MICROSOFT FORTRAN
C
C     DATA MCHEPS(1), MCHEPS(2) / #00000000, #3CB00000 /
C     DATA MINMAG(1), MINMAG(2) / #00000000, #00100000 /
C     DATA MAXMAG(1), MAXMAG(2) / #FFFFFFFF, #7FEFFFFF /
C
C     MACHINE CONSTANTS FOR THE IBM PC - PROFESSIONAL FORTRAN,
C     LAHEY FORTRAN, AND RM FORTRAN
C
C     DATA MCHEPS(1), MCHEPS(2) / Z'00000000', Z'3CB00000' /
C     DATA MINMAG(1), MINMAG(2) / Z'00000000', Z'00100000' /
C     DATA MAXMAG(1), MAXMAG(2) / Z'FFFFFFFF', Z'7FEFFFFF' /
C
C     MACHINE CONSTANTS FOR THE PDP-10 (KA PROCESSOR).
C
C     DATA MCHEPS(1),MCHEPS(2) / "114400000000, "000000000000 /
C     DATA MINMAG(1),MINMAG(2) / "033400000000, "000000000000 /
C     DATA MAXMAG(1),MAXMAG(2) / "377777777777, "344777777777 /
C
C     MACHINE CONSTANTS FOR THE PDP-10 (KI PROCESSOR).
C
C     DATA MCHEPS(1),MCHEPS(2) / "104400000000, "000000000000 /
C     DATA MINMAG(1),MINMAG(2) / "000400000000, "000000000000 /
C     DATA MAXMAG(1),MAXMAG(2) / "377777777777, "377777777777 /
C
C     MACHINE CONSTANTS FOR THE PDP-11 FORTRAN SUPPORTING
C     32-BIT INTEGERS (EXPRESSED IN INTEGER AND OCTAL).
C
C     DATA MCHEPS(1),MCHEPS(2) /  620756992,           0 /
C     DATA MINMAG(1),MINMAG(2) /    8388608,           0 /
C     DATA MAXMAG(1),MAXMAG(2) / 2147483647,          -1 /
C
C     DATA MCHEPS(1),MCHEPS(2) / O04470000000, O00000000000 /
C     DATA MINMAG(1),MINMAG(2) / O00040000000, O00000000000 /
C     DATA MAXMAG(1),MAXMAG(2) / O17777777777, O37777777777 /
C
C     MACHINE CONSTANTS FOR THE PDP-11 FORTRAN SUPPORTING
C     16-BIT INTEGERS (EXPRESSED IN INTEGER AND OCTAL).
C
C     DATA MCHEPS(1),MCHEPS(2) /   9472,      0 /
C     DATA MCHEPS(3),MCHEPS(4) /      0,      0 /
C
C     DATA MINMAG(1),MINMAG(2) /    128,      0 /
C     DATA MINMAG(3),MINMAG(4) /      0,      0 /
C
C     DATA MAXMAG(1),MAXMAG(2) /  32767,     -1 /
C     DATA MAXMAG(3),MAXMAG(4) /     -1,     -1 /
C
C     DATA MCHEPS(1),MCHEPS(2) / O022400, O000000 /
C     DATA MCHEPS(3),MCHEPS(4) / O000000, O000000 /
C
C     DATA MINMAG(1),MINMAG(2) / O000200, O000000 /
C     DATA MINMAG(3),MINMAG(4) / O000000, O000000 /
C
C     DATA MAXMAG(1),MAXMAG(2) / O077777, O177777 /
C     DATA MAXMAG(3),MAXMAG(4) / O177777, O177777 /
C
C     MACHINE CONSTANTS FOR THE UNIVAC 1100 SERIES.
C
C     DATA MCHEPS(1),MCHEPS(2) / O170640000000, O000000000000 /
C     DATA MINMAG(1),MINMAG(2) / O000040000000, O000000000000 /
C     DATA MAXMAG(1),MAXMAG(2) / O377777777777, O777777777777 /
C
```

```
C     MACHINE CONSTANTS FOR DEC VAX 11/780
C     (EXPRESSED IN INTEGER AND HEXADECIMAL)
C
C     DATA MCHEPS(1), MCHEPS(2) /      9472,          0 /
C     DATA MINMAG(1), MINMAG(2) /       128,          0 /
C     DATA MAXMAG(1), MAXMAG(2) /    -32769,         -1 /
C
C     DATA MCHEPS(1), MCHEPS(2) / Z00002500, Z00000000 /
C     DATA MINMAG(1), MINMAG(2) / Z00000080, Z00000000 /
C     DATA MAXMAG(1), MAXMAG(2) / ZFFFF7FFF, ZFFFFFFFF /
C
      DPMPAR = DMACH(I)
      RETURN
C
C     LAST CARD OF FUNCTION DPMPAR.
C
      END
```

The final value of the orbit gives the approximate solution

$$sin(7.49767761 - i \cdot 2.76867798) = 7.49767624 - i \cdot 2.76867834 \quad (1.16)$$

References

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ON USING DIFFERENTIAL EQUATIONS TO INVERT INTEGRAL EQUATIONS DESCRIBING ELECTROMAGNETIC SCATTERING BY HETEROGENEOUS BODIES

D. K. Cohoon
West Chester University

March 7, 1992

We are interested in predicting the scattering of electromagnetic radiation by heterogeneous structures. We can represent the electromagnetic fields induced within such a body as the solution of a coupled system of integral equation relating the electric and magnetic vectors of these fields to the electric and magnetic vectors of the stimulating electromagnetic field. The ideas developed here can be applied to bianisotropic structures, but for simplicity we restrict our attention to the case of a nonmagnetic body. By solving a differential equation, we develop a new inverse integral equation where only known functions appear under the integrals.

1 INTRODUCTION

When a scattering body has a general shape, there is no exact solution to the boundary value problem associated with Maxwell's equations. The problem is usually formulated in terms of integral equations where the field quantities $\vec{E}$ and $\vec{H}$ being sought appear both under the integral and outside the integral. The electric field integral equation has the form,

$$
\begin{aligned}
&\vec{E} - \vec{E}^{i} = \\
&-grad\left(\int_{\Omega}\frac{div(i\omega\epsilon\vec{E} + \alpha\vec{H} - i\omega\epsilon_0\vec{E})}{\omega\epsilon_0}G(r,s)dv(s)\right) \\
&+\frac{i}{\omega\epsilon_0}grad\left(\int_{\partial\Omega}(i\omega\epsilon\vec{E} + \alpha\vec{H} - i\omega\epsilon_0\vec{E})\cdot \boldsymbol{n}G(r,s)da(s)\right) \\
&+\left(\frac{-i}{\omega\epsilon_0}\right)\left(\int_{\partial\Omega}\left\{k_0^2(\boldsymbol{\sigma}_s(\vec{E} - (\vec{n}\cdot\vec{E})\vec{n}))G(r,s) + \right.\right. \\
&\left.\left. div(\boldsymbol{\sigma}_s(\vec{E} - (\vec{n}\cdot\vec{E})\vec{n}))grad(G(r,s))\right\} da(s)\right) \\
&-i\omega\mu_0\int_{\Omega}(i\omega\epsilon\vec{E} + \alpha\vec{H} - i\omega\epsilon_0\vec{E})G(r,s)dv(s) + \\
&-curl\left(\int_{\Omega}(i\omega\mu\vec{H} + \beta\vec{E} - i\omega\mu_0\vec{H})G(r,s)dv(s)\right) \qquad (1.1)
\end{aligned}
$$

and the magnetic field integral equation for a bianisotropic material is given by

$$
\begin{aligned}
&\vec{H} - \vec{H}^{i} = \\
&-grad\left(\int_{\Omega}\frac{div(i\omega\mu\vec{H} + \beta\vec{E} - i\omega\mu_0\vec{H})}{\omega\mu_0}G(r,s)dv(s)\right) \\
&-\frac{i}{\omega\mu_0}grad\int_{\partial\Omega}((i\omega\mu\vec{H} + \beta\vec{E} - i\omega\mu_0\vec{H})\cdot\boldsymbol{n})G(r,s)da(s) \\
&-\left(\int_{\partial\Omega}(\boldsymbol{\sigma}_s(\vec{E} - (\vec{n}\cdot\vec{E})\vec{n}) \times (grad(G(r,s)))da(s)\right) \\
&-i\omega\epsilon_0\int_{\Omega}(i\omega\mu\vec{H} + \beta\vec{E} - i\omega\mu_0\vec{H})G(r,s)dv(s) + \\
&+curl\left(\int_{\Omega}(i\omega\epsilon\vec{E} + \alpha\vec{H} - i\omega\epsilon_0\vec{E})G(r,s)dv(s)\right) \qquad (1.2)
\end{aligned}
$$

where ϵ, σ, μ, α, and β are tensors and the Maxwell equations for time harmonic radiation with an $exp(i\omega t)$ time dependence are given by

$$curl(\vec{E}) = i\omega\mu_0\vec{H} - \vec{J}_m \tag{1.3}$$

and

$$curl(\vec{H}) = i\omega\epsilon_0\vec{E} + \vec{J}_e \tag{1.4}$$

where

$$\vec{J}_e = i\omega\epsilon\vec{E} + \alpha\vec{H} - i\omega\epsilon_0\vec{E} \tag{1.5}$$

and

$$\vec{J}_m = i\omega\mu\vec{H} + \beta\vec{E} - i\omega\mu_0\vec{H} \tag{1.6}$$

To simplify the development we assume that the integral equation that we are solving has the form,

$$\vec{E} - \vec{E}^i = \lambda L\vec{E} \tag{1.7}$$

By working with this equation we have developed a resolvent operator $\mathcal{R}_\lambda$ such that

$$\vec{E}^i - \vec{E} = -\lambda R_\lambda \vec{E}^i \tag{1.8}$$

This resolvent operator R_λ is given by

$$R_\lambda \vec{E}^i(p) = \int_\Omega \mathcal{R}_\lambda(p,q)\vec{E}^i(q)d\nu(q) \tag{1.9}$$

and $\mathcal{R}_\lambda$ is the solution of the ordinary differential equation, in the independent variable λ, given by

$$\frac{d\mathcal{R}_\lambda}{d\lambda}(p,q) = \int_\Omega \mathcal{R}_\lambda(p,w)\mathcal{R}_\lambda(w,q)d\nu(w) \tag{1.10}$$

with

$$\mathcal{R}_0(p,q) = \mathcal{G}(p,q) \tag{1.11}$$

where

$$L\vec{E}(p) = \int_\Omega \mathcal{G}(p,q)\vec{E}(q)d\nu(q) \tag{1.12}$$

We note that once $\mathcal{R}_\lambda$ is known, we can predict the interaction of radiation with different orientations of the scattering body simply by applying transformations to $\vec{E}^i$ and calculating $\vec{E}$ for each of the transformed values of $\vec{E}^i$. This method gives us a kind of homotopy between the scattering problem for a vacuous scatterer to the more complex scattering problem.

2 OPERATOR ITERATES

The main theorem of Calderon and Zygmund ([2]) shows that if we define an operator L on the space $L^2(\Omega, \mathbb{C}^3)$ using the free space Green's function $\mathcal{G}$ by the rule,

$$L\vec{F}(p) = \int_\Omega \vec{F}(q)\mathcal{G}(p, q)d\nu(q), \tag{2.1}$$

then the operator norm of L is finite. The theorem of Calderon and Zygmund ([2]) tells us that the integral operators of electromagnetic scattering transform fields producing a finite total power into other fields producing a finite total power. Since all ℓ_p norms on $\mathbb{R}^n$ are equivalent, we may define the norm of L to be

$$| L |_\Omega = \ sup\{| Lf |_\Omega : f \epsilon L_2(\Omega, \mathbb{C}^3), \quad and \, | f |_\Omega = 1\} \tag{2.2}$$

where

$$f(p) = (f_1(p), f_2(p), f_3(p)) \tag{2.3}$$

implies that

$$| f |_\Omega^2 = \sum_{i=1}^{3} \int_\Omega | f_i(y) |^2 \, d\nu(y) \tag{2.4}$$

It is clear, therefore, that if λ is sufficiently small that the operator norm of λL is smaller than 1. Thus, in everything that follows in this section we shall assume that

$$| \lambda L |_\Omega < 1 \tag{2.5}$$

It is now easy to derive an expression for $\vec{E}^i - \vec{E}_\lambda$ under this assumption. Just using concepts associated with the summation of a geometric series we find that

$$\vec{E}^i - \vec{E}_\lambda = -\lambda \left(\sum_{k=1}^{\infty} \lambda^{k-1} L^k \vec{E}^i \right) \tag{2.6}$$

We express the right side of equation (2.6) as an integral operator by introducing the sought after solution finder or resolvent kernel $\mathcal{R}_\lambda(p, q)$ via the relationship

$$-\lambda \int_\Omega \mathcal{R}_\lambda(p, q) \cdot \vec{E}^i(q) d\nu(q) = -\lambda \sum_{k=1}^{\infty} \left(\lambda^{k-1} L^k \vec{E}^i \right) \tag{2.7}$$

Combining (2.6) and (2.7) and the basic definition (2.1) of L in terms of $\mathcal{G}(p,q)$ imply that if we introduce the functions $\mathcal{G}^{(k)}(p,q)$ by the relations,

$$\mathcal{R}_\lambda(p,q) = \mathcal{G}(p,q) + \sum_{k=1}^{\infty} \left(\lambda^k \mathcal{G}^{(k+1)}(p,q)\right) \tag{2.8}$$

so that it would then follow that

$$L^k \vec{E}^i = \int_\Omega \mathcal{G}^{(k)}(p,q)\vec{E}^i(q)d\nu(q) \tag{2.9}$$

and since

$$L^{k+1}\vec{E}^i = \int_\Omega \mathcal{G}(p,w)\left(\int_\Omega \mathcal{G}^{(k)}(w,q)\vec{E}^i(q)d\nu(q)\right)d\nu(w) \tag{2.10}$$

and since an interchange of the order of integration in (2.10) implies that in view of (2.9) and the relationship,

$$L^{k+1}\vec{E}^i = L(L^k\vec{E}^i) = L^k(L\vec{E}^i) \tag{2.11}$$

that

$$\begin{aligned}\mathcal{G}^{(k+1)}(p,q) &= \int_\Omega \mathcal{G}(p,w)\mathcal{G}^{(k)}(w,q)d\nu(w) \\ &= \int_\Omega \mathcal{G}^{(k)}(p,w)\mathcal{G}(w,q)d\nu(w)\end{aligned} \tag{2.12}$$

it will follow upon substitution of (2.12) into (2.8) that

$$\mathcal{R}_\lambda(p,q) = \mathcal{G}(p,q) + \sum_{k=1}^{\infty} \lambda^k \left(\int_\Omega \mathcal{G}(p,w)\mathcal{G}^{(k)}(w,q)d\nu(w)\right) \tag{2.13}$$

We now resubstitute the original representation of $\mathcal{R}_\lambda$ given by (2.8) into (2.12) making use of the fact that

$$\mathcal{G}^{(1)}(p,q) = \mathcal{G}(p,q)$$

to deduce that since (2.8) says that

$$\lambda\mathcal{R}_\lambda(p,q) = \sum_{k=1}^{\infty}\lambda^k\mathcal{G}^{(k)}(p,q) =$$

$$\lambda\left(\mathcal{G}(p,q) + \sum_{k=1}^{\infty}\lambda^k\mathcal{G}^{(k+1)}(p,q)\right), \tag{2.14}$$

and since

$$\sum_{k=1}^{\infty}\lambda^k\mathcal{G}^{(k+1)}(p,q) = \sum_{k=1}^{\infty}\left(\int_\Omega \mathcal{G}(p,w)\mathcal{G}^{(k)}(w,q)d\nu(w)\right) =$$

$$\int_\Omega \mathcal{G}(p,w)\left(\sum_{k=1}^{\infty}\lambda^k\mathcal{G}^{(k)}(w,q)\right)d\nu(w) = \int_\Omega \mathcal{G}(p,w)\lambda\mathcal{R}_\lambda(w,q)d\nu(w) \tag{2.15}$$

the relation,

$$\mathcal{R}_\lambda(p,q) = \mathcal{G}(p,q) + \lambda\left(\int_\Omega \mathcal{G}(p,w)\mathcal{R}_\lambda(w,q)d\nu(w)\right) \tag{2.16}$$

is valid for λ with a sufficiently small absolute value. Our next objective is to obtain an expression for

$$\mathcal{Q}(\lambda,\tilde{\lambda}) = \left(\frac{\mathcal{R}_{\tilde{\lambda}} - \mathcal{R}_\lambda}{\tilde{\lambda} - \lambda}\right) \tag{2.17}$$

and take the limit as $\tilde{\lambda}$ approaches λ.

We begin by noticing that in view of equations (2.12) (2.14), and (2.16), we obtain the relation,

$$\lambda\int_\Omega \mathcal{R}_\lambda(p,w)\mathcal{G}(w,q)d\nu(w) - \tilde{\lambda}\int_\Omega \mathcal{G}(p,w)\mathcal{R}_{\tilde{\lambda}}(w,q)d\nu(w) =$$

$$\int_\Omega\Bigg\{\sum_{j=0}^{\infty}\left(\lambda^{j+1}\mathcal{G}^{(j+1)}(p,w)\mathcal{G}(w,q)\right) -$$

$$\sum_{j=0}^{\infty}\left(\tilde{\lambda}^{j+1}\mathcal{G}^{(j+1)}(w,q)\mathcal{G}(p,w)\right)\Bigg\}d\nu(w) \tag{2.18}$$

In working with equation (2.18) we will make use of the standard identity

$$\lambda^{j+1} - \tilde{\lambda}^{j+1} = (\lambda - \tilde{\lambda})\left(\sum_{k=0}^{j}\left(\lambda^k\tilde{\lambda}^{j-k}\right)\right) \tag{2.19}$$

and the fact that (2.12) implies that

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\left(\lambda^k\bar{\lambda}^{j-k}\mathcal{G}^{(j+2)}(p,q)\right)\right) =$$

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\left(\lambda^k\bar{\lambda}^{(j-k)}\right)\left(\int_\Omega \mathcal{G}(p,w)\mathcal{G}^{(j+1)}(w,q)d\nu(w)\right)\right) \tag{2.20}$$

However, in order to proceed we need the following Lemma.

Lemma 2.1 *If $\mathcal{G}$ is a dyadic Calderon-Zygmund kernel (Calderon and Zygmund [2]) on the open set Ω of $\mathbb{R}^n$ and if $\mathcal{G}^{(k)}$ is defined by equation (2.12), then if j is a nonnegative integer,*

$$\int_\Omega \mathcal{G}(p,w)\mathcal{G}^{(j+1)}(w,q)d\nu(q) =$$

$$\int_\Omega \mathcal{G}^{(k+1)}(p,w)\mathcal{G}^{(j-k+1)}(w,q)d\nu(q) \tag{2.21}$$

for all integers k between 0 and j.

Proof of Lemma 2.1. The proof of the Lemma will proceed by induction on j. If $j=0$, then $k=0$ and equation (2.21) is a tautology. Thus, we let $\mathcal{P}(j+1)$ be the sentence that says that equation (eq: iteratedintSGS-Gsupjplus1) is valid for the nonnegative integer j. We have just observed that $\mathcal{P}(1)$ is true, and we proceed to prove that $\mathcal{P}(n)$ implies that $\mathcal{P}(n+1)$ is true. We note that $\mathcal{P}(n)$ is always true if $k=0$ or if $k=j$, and we consequently assume that $0<k<j$ and proceed by induction on j. The definition of $\mathcal{G}^{(j+1)}(w,q)$ and the inductive hypothesis imply that

$$\int_\Omega \mathcal{G}(p,w)\mathcal{G}^{(j+1)}(w,q)d\nu(w) =$$

$$\int_\Omega \mathcal{G}(p,w)\int_\Omega\left(\mathcal{G}(w,u)\mathcal{G}^{(j)}(u,q)\right)d\nu(u)d\nu(w) =$$

$$\int_\Omega \mathcal{G}(p,w)\int_\Omega \mathcal{G}^{(k-1)+1)}(w,u)\mathcal{G}^{(j-1)-(k-1)+1}(u,q)d\nu(q)d\nu(w) \tag{2.22}$$

Interchanging the order of integration in equation (2.22) implies that

$$\int_\Omega \mathcal{G}(p,w)\mathcal{G}^{(j+1)}(w,q)d\nu(w) =$$

$$\int_\Omega \left(\int_\Omega \mathcal{G}^{(k-1)+1}(w,u)\right)\mathcal{G}^{(j-k+1)}(u,q)d\nu(u)$$

$$= \int_\Omega \mathcal{G}^{(k+1)}(p,u)\mathcal{G}^{(j-k+1)}(u,q)d\nu(u) \tag{2.23}$$

and this completes the proof of Lemma 2.1.

We will now use Lemma 2.1, equation (2.21), to rewrite equation (eq: scriptGrecursion) in the form

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\lambda^k\bar{\lambda}^{j-k}\mathcal{G}^{(j+2)}(p,q)\right) =$$

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\left(\lambda^k\bar{\lambda}^{j-k}\int_\Omega \mathcal{G}^{(k+1)}(p,w)\mathcal{G}^{(j-k+1)}(w,q)d\nu(w)\right)\right) \tag{2.24}$$

We will now prove the validity of another Lemma. This Lemma will be more abstract and will treat properties of sequences of, possibly, noncommuting linear transformations $\{A_1, A_2, A_3, \cdots\}$ and $\{B_1, B_2, B_3, \cdots\}$ where the A_k map a Banach space Y onto a Banach space Z, and the B_j map a Banach space X onto the Banach space Y, and the conditions under which one may define the product of a series of the form,

$$\mathcal{A} = \sum_{k=0}^{\infty}(\alpha_k A_{k+1})$$

and a series of the form

$$\mathcal{B} = \sum_{j=0}^{\infty}(\beta_j B_{j+1})$$

While the Lemma which follows may appear to be formally obvious, a proof is needed because of the interchange of infinite processes.

The Lemma is the following.

Lemma 2.2 *Let $\{A_1, A_2, A_3, \cdots\}$ be a sequence of bounded linear transformations of the Banach space Y with norm, $|\ \ |_Y$ into the Banach space Z with norm, $|\ \ |_Z$. Let $\{B_1, B_2, B_3, \cdots\}$ be a sequence of bounded linear*

transformations of the Banach space X with norm, $|\ |_X$, *into the Banach space Y such that if*

$$| A_j |_{(Y,Z)} = sup\{| A_j f |_Z:\ f \in Y \text{ and } | f |_Y = 1\}$$

and

$$| B_k |_{(X,Y)} = sup\{| B_k f |_Y:\ f \in X \text{ and } | f |_X = 1\}$$

then there are positive real constants C_A, C_B, R_A, *and* R_B *with the property that*

$$| A_{k+1} |_{(Y,Z)} \leq C_A R_A^k$$

and

$$| B_{j+1} |_{(X,Y)} \leq C_B R_B^j$$

for

$$\{j, k\} \subset \{1, 2, 3, \cdots\}.$$

If λ *and* $\tilde{\lambda}$ *are such that* $\lambda R_A < 1$ *and* $\tilde{\lambda} R_B < 1$, *then*

$$\left(\sum_{k=0}^{\infty}\left(\lambda^k A_{k+1}\right)\right)\left(\sum_{j=0}^{\infty}\left(\tilde{\lambda}^j B_{j+1}\right)\right) =$$

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\left(\lambda^k \tilde{\lambda}^{j-k} A_{k+1} B_{j+1-k}\right)\right) \tag{2.25}$$

and either side of this equation represents a bounded linear transformation of the Banach space X into the Banach space Z.

Proof of Lemma 2.2. Since B_{j+1-k} maps X into Y and A_{k+1} maps Y into Z, it is clear that $A_{k+1}B_{j+1-k}$ transforms elements of X linearly and continuously into Z. Also, the hypothesis of Lemma 2.2 guarantee that both sides of equation (2.25) define uniformly series of bounded linear operators acting on the Banach space X and that, consequently, any rearrangement of terms leaves the sum unchanged. Since

$$\lambda^k \tilde{\lambda}^\ell = \lambda^k \tilde{\lambda}^{j-k}$$

if $k + \ell = j$, the Lemma follows by induction on the products of the number of terms in finite partial sums approximating the left side of equation (2.25).

We now apply Lemma 2.2 to prove the following.

Lemma 2.3 *If $\mathcal{R}_\lambda$ is defined for complex numbers λ by equation (2.8) or by equation (eq: lambdaRsublambda), then the relationship (2.12) represents $\mathcal{G}^{(j+2)}$ and*

$$\int_\Omega \mathcal{R}_\lambda(p,w)\mathcal{R}_{\bar{\lambda}}(w,q)d\nu(w) =$$

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\left(\lambda^k\bar{\lambda}^{j-k}\mathcal{G}^{(j+2)}(p,q)\right)\right) \tag{2.26}$$

Proof of Lemma 2.3. By equation (eq: Rsublambda) we see that

$$\mathcal{R}_\lambda(p,w) = \sum_{k=0}^{\infty}\left(\lambda^k\mathcal{G}^{(k+1)}(p,w)\right) \tag{2.27}$$

Thus, by Lemma 2.2 it follows that

$$\int_\Omega \mathcal{R}_\lambda(p,w)\mathcal{R}_{\bar{\lambda}}(w,q)d\nu(w)$$

$$\int_\Omega\left(\sum_{k=0}^{\infty}\left(\lambda^k\mathcal{G}^{(k+1)}(p,w)\right)\right)\left(\sum_{j=0}^{\infty}\left(\bar{\lambda}^j\mathcal{G}^{(j+1)}(w,q)\right)\right)d\nu(w) =$$

$$\int_\Omega\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\left(\lambda^k\bar{\lambda}^{j-k}\mathcal{G}^{(k+1)}(p,w)\mathcal{G}^{(j-k+1)}(w,q)\right)\right)\right)d\nu(w) \tag{2.28}$$

Now using the relation (2.22) and the definition, equation (eq: Gsupkplusoneofpandq), of $\mathcal{G}^{(j+1)}(p,q)$ we see that

$$\mathcal{G}^{(j+2)}(p,q) = \int_\Omega \mathcal{G}^{(k+1)}(p,w)\mathcal{G}^{(j-k+1)}(w,q)d\nu(w) \tag{2.29}$$

Thus, Lemma 2.3 and equation (2.26) then follows as a result of substituting equation (2.29) into equation (2.28). This completes the proof of Lemma 2.3.

We now complete the proof of the final Lemma which will give us an expression for

$$f(\lambda,\bar{\lambda}) = \mathcal{R}_\lambda(p,q) - \mathcal{R}_{\bar{\lambda}}(p,q)$$

Equation (2.16) then tells us that

$$\mathcal{R}_\lambda(p,q) - \mathcal{R}_{\bar{\lambda}}(p,q) =$$

$$\lambda \int_\Omega \mathcal{G}(p,w)\mathcal{R}_\lambda(w,q)d\nu(w) - \tilde{\lambda}\int_\Omega \mathcal{G}(p,w)\mathcal{R}_{\tilde{\lambda}}(w,q)d\nu(w) \qquad (2.30)$$

Substitution of the power series representation, equation (2.8), of $\mathcal{R}_\lambda(w,q)$ into (2.30), we can obtain the relationship,

$$\mathcal{R}_\lambda(p,q) - \mathcal{R}_{\tilde{\lambda}}(p,q) =$$

$$\sum_{j=0}^{\infty}\left(\lambda^{j+1}\mathcal{G}^{(j+2)}(p,q) - \tilde{\lambda}^{j+1}\mathcal{G}^{(j+2)}(p,q)\right) =$$

$$\lambda \int_\Omega \mathcal{G}(p,w)\left(\sum_{j=0}^{\infty}\left(\lambda^j\mathcal{G}^{(j+1)}(w,q)\right)\right)d\nu(w) -$$

$$\tilde{\lambda}\int_\Omega \mathcal{G}(p,w)\left(\sum_{j=0}^{\infty}\left(\tilde{\lambda}^j\mathcal{G}^{(j+1)}(w,q)\right)\right)d\nu(w) \qquad (2.31)$$

Substituting (2.19) into equation (2.31) gives us the following lemma

Lemma 2.4 . *If $\mathcal{R}_\lambda(p,q)$ is given by equation (2.8), where $\mathcal{G}^{(k)}(p,q)$ is defined by (2.12), then*

$$\mathcal{R}_\lambda(p,q) - \mathcal{R}_{\tilde{\lambda}}(p,q) =$$

$$(\lambda - \tilde{\lambda})\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\left(\lambda^k\tilde{\lambda}^{j-k}\mathcal{G}^{(j+2)}(p,q)\right)\right)\right) \qquad (2.32)$$

These Lemmas enable one to prove the following theorem.

Theorem 2.1 *If $\mathcal{R}_\lambda(p,q)$ is defined by (2.7) and (2.8), and $\mathcal{G}(p,q)$ is a Calderon Zygmund Kernel (Calderon and Zygmund [2]), then*

$$\mathcal{R}_\lambda(p,q) - \mathcal{R}_{\tilde{\lambda}}(p,q) =$$

$$(\lambda - \tilde{\lambda})\int_\Omega \mathcal{R}_\lambda(p,w)\mathcal{R}_{\tilde{\lambda}}(w,q)d\nu(w) \qquad (2.33)$$

and

$$\frac{d\mathcal{R}_\lambda}{d\lambda}(p,q) = \int_\Omega \mathcal{R}_\lambda(p,w)\mathcal{R}_\lambda(w,q)d\nu(w) \qquad (2.34)$$

where

$$\mathcal{R}_0(p,q) = \mathcal{G}(p,q) \qquad (2.35)$$

Proof of Theorem 2.1. Equation (2.33) follows by substituting equation (2.26) into (2.31). Equation (2.34) follows by dividing both sides of (2.33) by $\lambda - \tilde{\lambda}$ and taking the limit as $\tilde{\lambda}$ approaches λ. Equation (2.35) follows from equation (2.16).

In solving the initial value problem suggested by this theorem we note that the Cauchy integral theorem tells us that an integral of $\mathcal{R}_\lambda$ over a curve or a path of λ values in the complex plane is independent of path if one path can be deformed into another without crossing a pole of $\mathcal{R}_\lambda$.

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UNIQUENESS OF SOLUTIONS OF ELECTROMAGNETIC INTERACTION PROBLEMS ASSOCIATED WITH SCATTERING BY BIANISOTROPIC BODIES COVERED WITH IMPEDANCE SHEETS

D. K. Cohoon

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Bianisotropic materials are more general than either anisotropic or chiral materials. We write down the frequency domain Maxwell equations for a bianisotropic material and develop conditions on tensors appearing in these equations which guarantee uniqueness of the solution of the electromagnetic interaction problem. The primary tools here are the the use of Silver Mueller radiation conditions identities involving integrals of field quantities over the interior and surface of the scattering body derived from impedance sheet boundary conditions, and some theorems of Gauss. The integral equation formulation of the electromagnetic interaction problem is provided in three and seven dimensional space.

1 INTRODUCTION

Bianisotropic materials(Kong [7]), because of their greater complexity, have greater potential for reliably modeling physical materials which respond linearly to stimulating electromagnetic radiation. Chiral properties are a special case of bianisotropic materials. With chiral materials there is a special scalar ξ_c (Jaggard and Engetta [6]) such that

$$\vec{D} = \epsilon\vec{E} + i\xi_c\vec{B} \tag{1.1}$$

and

$$\vec{B} = \mu\vec{H} - i\xi_c\mu\vec{E} \tag{1.2}$$

With the more general bianisotropic materials there are tensors α and β with the property that

$$\vec{D} = \epsilon\vec{E} + \alpha\vec{H}/(i\omega) \tag{1.3}$$

and

$$\vec{B} = \mu\vec{H} + \beta\vec{E}/(i\omega) \tag{1.4}$$

where ϵ and μ are tensors. Here Maxwell's equations have the form

$$curl(\vec{E}) = -i\omega\vec{B} \tag{1.5}$$

and

$$curl(\vec{H}) = i\omega\vec{D} + \sigma\vec{E}. \tag{1.6}$$

Using these notions we make Maxwell's equations look like the standard Maxwell equations with complex sources by introducing the generalized electric and magnetic current densities by the relations,

$$curl(\vec{E}) = i\omega\mu_0\vec{H} - \vec{J}_m \tag{1.7}$$

and

$$curl(\vec{H}) = i\omega\epsilon_0\vec{E} + \vec{J}_e \tag{1.8}$$

where

$$\vec{J}_e = i\omega\epsilon\vec{E} + \alpha\vec{H} - i\omega\epsilon_0\vec{E} \tag{1.9}$$

and

$$\vec{J}_m = i\omega\mu\vec{H} + \beta\vec{E} - i\omega\mu_0\vec{H} \tag{1.10}$$

We also assume that there is an impedance sheet current density given by

$$\vec{J}_s = \sigma_s(\vec{E} - (\vec{n}\cdot\vec{E})\vec{n}) \tag{1.11}$$

The formulation of integral equations for bianisotropic materials, therefore, is carried out by the analysis of the following coupled system of integral equations based on the notion of electric and magnetic charges defined by the two continuity equations

$$div(\vec{J}_e) + \frac{\partial \rho_e}{\partial t} \tag{1.12}$$

and

$$div(\vec{J}_m) + \frac{\partial \rho_m}{\partial t} \tag{1.13}$$

Having developed this the coupled system of integral equations describing the interaction of electromagnetic radiation with a bounded bianisotropic body Ω is given by the following relations. The electric field integral equation is given by

$$\begin{aligned}\vec{E} - \vec{E}^i = &-grad\left(\int_\Omega \frac{div(\vec{J}_e)}{\omega\epsilon_0} G(r,s)dv(s)\right)\\ &+\frac{i}{\omega\epsilon_0} grad\left(\int_{\partial\Omega}(\vec{J}_e \cdot n)G(r,s)da(s)\right)\\ &+\left(\frac{-i}{\omega\epsilon_0}\right)\left(\int_{\partial\Omega}\left\{k_0^2(\vec{J}_s)G(r,s) + div(\vec{J}_s)grad(G(r,s))\right\}da(s)\right)\\ &-i\omega\mu_0\int_\Omega \vec{J}_e G(r,s)dv(s)+\\ &-curl\left(\int_\Omega \vec{J}_m G(r,s)dv(s)\right)\end{aligned} \tag{1.14}$$

and the magnetic field integral equation may be expressed as

$$\vec{H} - \vec{H}^i = -grad\left(\int_\Omega \frac{div(\vec{J}_m)}{\omega\mu_0} G(r,s)dv(s)\right)$$

$$-\frac{i}{\omega\mu_0} grad\left(\int_{\partial\Omega} (\vec{J}_m \cdot n)G(r,s)da(s)\right)$$

$$-\left(\int_{\partial\Omega} (\vec{J}_s \times (grad(G(r,s)))da(s)\right)$$

$$-i\omega\epsilon_0 \int_\Omega \vec{J}_m G(r,s)dv(s)+$$

$$+ curl\left(\int_\Omega \vec{J}_e G(r,s)dv(s)\right) \tag{1.15}$$

where G(r,s) is the rotation invariant, temperate fundamental solution of the Helmholtz equation,

$$(\Delta + k_0^2)G = \delta \tag{1.16}$$

given by

$$G(r,s) = \frac{exp(-ik_0 \mid r - s \mid)}{4\pi \mid r - s \mid} \tag{1.17}$$

Substituting (1.9) through (1.11) into equations (1.14) and (1.15) we obtain, the coupled integral equations for bianisotropic materials. The electric field integral equation for a bianisotropic material is given by,

$$\vec{E} - \vec{E}^i = -grad\left(\int_\Omega \frac{div(i\omega\epsilon\vec{E} + \alpha\vec{H} - i\omega\epsilon_0\vec{E})}{\omega\epsilon_0} G(r,s)dv(s)\right)$$

$$+\frac{i}{\omega\epsilon_0} grad\left(\int_{\partial\Omega} (i\omega\epsilon\vec{E} + \alpha\vec{H} - i\omega\epsilon_0\vec{E} \cdot n)G(r,s)da(s)\right)$$

$$+\left(\frac{-i}{\omega\epsilon_0}\right)\left(\int_{\partial\Omega} \left\{k_0^2(\sigma_s(\vec{E} - (\vec{n}\cdot\vec{E})\vec{n}))G(r,s)+\right.\right.$$

$$\left.\left. div(c_s(\vec{E} - (\vec{n}\cdot\vec{E})\vec{n}))grad(G(r,s))\right\} da(s)\right)$$

$$-i\omega\mu_0 \int_\Omega (i\omega\epsilon\vec{E} + \alpha\vec{H} - i\omega\epsilon_0\vec{E})G(r,s)dv(s)+$$

$$- curl\left(\int_\Omega (i\omega\mu\vec{H} + \beta\vec{E} - i\omega\mu_0\vec{H})G(r,s)dv(s)\right) \tag{1.18}$$

The magnetic field integral equation for a bianisotropic material covered by an impedance sheet is given by

$$\vec{H} - \vec{H}^i = -grad\left(\int_\Omega \frac{div(i\omega\mu\vec{H} + \beta\vec{E} - i\omega\mu_0\vec{H})}{\omega\mu_0} G(r,s)dv(s)\right)$$

$$-\frac{i}{\omega\mu_0} grad \int_{\partial\Omega} (i\omega\mu\vec{H} + \beta\vec{E} - i\omega\mu_0\vec{H}\cdot n)G(r,s)da(s)$$

$$-\left(\int_{\partial\Omega}(\sigma_s(\vec{E} - (\vec{n}\cdot\vec{E})\vec{n}) \times (grad(G(r,s)))da(s)\right)$$

$$-i\omega\epsilon_0 \int_\Omega (i\omega\mu\vec{H} + \beta\vec{E} - i\omega\mu_0\vec{H})G(r,s)dv(s)+$$

$$+ curl\left(\int_\Omega (i\omega\epsilon\vec{E} + \alpha\vec{H} - i\omega\epsilon_0\vec{E})G(r,s)dv(s)\right) \qquad (1.19)$$

While we have obtained exact solutions for layered materials, most of the problems are so complex that one must formulate the interaction problems using integral equations. The primary focus of this paper is to demonstrate the equivalence of integral equation and Maxwell equation formulations of the problem for suitable function spaces by demonstrating uniqueness. Then we can carry out the design of complex materials using an improvement of classical spline methods (Tsai, Massoudi, Durney, and Iskander [9], pp 1131-1139) and (Li [8]). The Tsai, Massoudi, Durney, and Iskander paper is unusual in that comparisons are made between internal fields predicted from moment method computations and Mie solution computations. In Li [8] this verification was carried out analytically. There are several methods of solving coupled integral equations of the form (1.18) and (1.19) by approximate methods. However, as the scattering bodies become more complex the computational requirements become larger and larger. With exact finite rank integral equation theory ([3]), if one has a discretization that enables one to closely approximate the solution, then refinements can be made by a process based on the concept that the norm of the difference between an approximate integral operator and the actual integral operator is simply smaller than one, not necessarily close enough to give answers of acceptable accuracy. Then the answer is improved by an iterative process to any desired precision without the use of additional computer memory.

When solving electromagnetic scattering problems for isotropic bodies, one can proceed in a fairly direct way with the discretization of specializations of the integral equations (1.18) and (1.19), but in modeling classes of the more complex materials such as liquid crystals for electron camera shutters or piezoelectric materials for micromotors, greater care has to be taken that the model of the interaction has only one solution.

2 UNIQUENESS

If $\vec{E}$ and $\vec{H}$ are electric and magnetic fields in a bianisotropic material, then there exist tensors μ, ϵ, α, and β such that

$$curl(\vec{E}) = -i\omega\mu\vec{H} - \beta\vec{E} \tag{2.1}$$

and

$$curl(\vec{H}) = (i\omega\varepsilon + \sigma)\vec{E} + \alpha\vec{H} \tag{2.2}$$

We assume that if α is a complex tensor that α^* denotes its complex conjugate. We assume that a bounded bianisotropic body Ω with a smooth normal is embedded in three dimensional free space and is subjected to a remote source of radiation whose electric field is $\vec{E}^i$ and whose magnetic field is $\vec{H}^i$. If $\vec{E}$ denotes the difference between two solutions of the form $\vec{E}^i + \vec{E}^s$, where $\vec{E}^s$ denotes the scattered radiation, in the exterior of Ω, or simply the difference of two solutions in the interior of Ω, then the solution is unique if we can prove that $\vec{E}$ is identically zero in the exterior of Ω or everywhere inside Ω.

The starting point for proofs of uniqueness of solutions of electromagnetic scattering problems is the Silver Mueller radiation conditions which demand that if C_R is a sphere of radius R centered at a point in the scattering body, that then

$$\lim_{R \to \infty} \int_{C_R} (\vec{n} \times curl(\vec{E}) - ik_0\vec{E}) \cdot (\vec{n} \times curl(\vec{E}^*) + ik_0\vec{E}^*) da = 0 \tag{2.3}$$

Thus, we note that

$$\int_{C_R} | \vec{n} \times curl(\vec{E}) - ik_0\vec{E} |^2 \, da =$$

$$\int_{C_R} (| \vec{n} \times curl(\vec{E}) |^2 + k_0^2 | E |^2) da +$$

$$ik_0 \int_{C_R} ((\vec{n} \times curl(\vec{E})) \cdot \vec{E}^*) da$$

$$- ik_0 \int_{C_R} \vec{E} \cdot (\vec{n} \times curl(\vec{E}^*)) da \tag{2.4}$$

Focusing our attention on the last two terms in this equation, we see that

$$ik_0 \int_{C_R} (\vec{n} \times curl(\vec{E})) \cdot \vec{E}^* da =$$

$$ik_0 \int_{C_R} \vec{n} \cdot (curl(\vec{E}) \times \vec{E}^*) dv$$

$$= ik_0 \int_{V_1} div(curl(\vec{E}) \times \vec{E}^*) dv +$$

$$+ (ik_0) \int_{S_2} \vec{n} \cdot (curl(\vec{E}) \times \vec{E}^*) da \tag{2.5}$$

In the previous equation S_2 is the surface bounding the bianisotropic body and V_1 is the region between the bounded bianisotropic body and the sphere C_R centered at a point in the bianisotropic material. We will assume that V_2 represents the bounded bianisotropic body covered by an impedance sheet. Continuing our analysis, and replacing $curl(\vec{E})$ by $-i\omega\mu_0\vec{H}$ we find that

$$ik_0 \int_{C_R} (\vec{n} \times curl(\vec{E})) \cdot \vec{E}^* da =$$

$$ik_0 \int_{C_R} div(curl(\vec{E}) \times \vec{E}^*) dv +$$

$$k_0\omega\mu_0 \int_{S_2} \vec{n} \cdot (\vec{H} \times \vec{E}^*) da. \tag{2.6}$$

Thus, making use of the impedance sheet boundary condition which states that

$$\vec{n} \times \vec{H} = \vec{n} \times \vec{H}_2 + \sigma_s(\vec{E}_2 - (\vec{n} \cdot \vec{E}_2)\vec{n}) \tag{2.7}$$

we find that

$$ik_0 \int_{C_R} (\vec{n} \times curl(\vec{E})) \cdot \vec{E}^* da =$$
$$ik_0 \int_{C_R} div(curl(\vec{E}) \times \vec{E}^*) dv +$$
$$k_0 \omega \mu_0 \int_{S_2} \vec{n} \cdot (\vec{H}_2 \times \vec{E}_2^*) da +$$
$$k_0 \omega \mu_0 \int_{S_2} \left\{ \sigma_s (\vec{E}_2 - (\vec{n} \cdot \vec{E}_2)\vec{n}) \cdot \vec{E}_2^* \right\} da \tag{2.8}$$

where $\vec{H}_2$ and $\vec{E}_2$ are the electric and magnetic fields just inside of the impedance sheet on the surface S_2. First using the Gauss divergence theorem we find that

$$ik_0 \int_{C_R} (\vec{n} \times curl(\vec{E})) \cdot \vec{E}^* da =$$
$$ik_0 \int_{C_R} div(curl(\vec{E}) \times \vec{E}^*) dv +$$
$$k_0 \omega \mu_0 \int_{V_2} div(\vec{H}_2 \times \vec{E}_2^*) dv +$$
$$k_0 \omega \mu_0 \int_{S_2} \left\{ \sigma_s (\vec{E}_2 - (\vec{n} \cdot \vec{E}_2)\vec{n}) \cdot \vec{E}_2^* \right\} da \tag{2.9}$$

We now make use of the vector calculus identity,

$$div(\vec{A} \times \vec{B}) = \vec{B} \cdot (curl(\vec{A})) - \vec{A} \cdot (curl(\vec{B})) \tag{2.10}$$

We find that

$$ik_0 \int_{C_R} (\vec{n} \times curl(\vec{E})) \cdot \vec{E}^* da =$$
$$ik_0 \int_{V_1} (\vec{E}^* \cdot (curl(curl(\vec{E}))) - curl(\vec{E}) \cdot curl(\vec{E}^*)) dv +$$
$$k_0 \omega \mu_0 \int_{V_2} (\vec{E}_2^* \cdot curl(\vec{H}_2) - \vec{H}_2 \cdot curl(\vec{E}_2^*)) dv +$$
$$k_0 \omega \mu_0 \int_{S_2} \sigma_s \left\{ (\vec{E}_2 \cdot \vec{E}_2^*) - (\vec{E}_2 \cdot \vec{n})(\vec{E}_2^* \cdot \vec{n}) \right\} da \tag{2.11}$$

Substituting in the constitutive relations we find that

$$ik_0 \int_{C_R} (\vec{n} \times curl(\vec{E})) \cdot \vec{E}^* da = -ik_0 \int_{V_1} \left\{ \vec{E}^* \cdot \Delta \vec{E} + \mid curl(\vec{E}) \mid^2 \right\} dv$$

$$+k_0 \omega \mu_0 \int_{V_2} \vec{E}_2^* \cdot \left\{ (i\omega\epsilon + \sigma)\vec{E}_2 + \alpha \vec{H}_2 \right\} dv$$

$$-k_0 \omega \mu_0 \int_{V_2} \vec{H}_2 \cdot (i\omega\mu^* \vec{H}_2^* - \beta^* \vec{E}_2^*) dv +$$

$$k_0 \omega \mu_0 \int_{S_2} \sigma_s \left\{ \mid (\vec{E}_2 - (\vec{E}_2 \cdot \vec{n}) \cdot \vec{n}) \mid^2 \right\} da \tag{2.12}$$

Considering the conjugate term of this form we observe that

$$-ik_0 \int_{C_R} (\vec{n} \times curl(\vec{E}^*)) \cdot \vec{E} da = ik_0 \int_{V_1} \left\{ \vec{E} \cdot \Delta \vec{E}^* + \mid curl(\vec{E}) \mid^2 \right\} dv$$

$$+k_0 \omega \mu_0 \int_{V_2} \vec{E}_2 \cdot \left\{ (-i\omega\epsilon^* + \sigma^*)\vec{E}_2^* + \alpha^* \vec{H}_2^* \right\} dv$$

$$-k_0 \omega \mu_0 \int_{V_2} \vec{H}_2^* \cdot (-i\omega\mu \vec{H}_2 - \beta \vec{E}_2) dv +$$

$$k_0 \omega \mu_0 \int_{S_2} \sigma_s^* \left\{ \mid (\vec{E}_2 - (\vec{E}_2 \cdot \vec{n}) \cdot \vec{n}) \mid^2 \right\} da \tag{2.13}$$

Adding these equations we find that the solution is unique provided that either a quadratic form is positive definite or another form is either negative or positive definite. Indeed it may be easier to prove uniqueness for the more complex material than when the scatterer has a simpler form. We find that

$$2Re\left(ik_0\int_{C_R}(\vec{n}\times curl(\vec{E}))\cdot\vec{E}^*da\right)=$$

$$k_0\omega\mu_0\int_{V_2}\left\{(\vec{E}_2^*\cdot(i\omega\epsilon+\sigma)\vec{E}_2)+(\vec{E}_2\cdot(-i\omega\epsilon^*+\sigma^*)\vec{E}_2^*)\right\}dv+$$

$$k_0\omega\mu_0\int_{V_2}\left\{(\vec{E}_2^*\cdot(\alpha)\vec{H}_2)+(\vec{E}_2\cdot(\alpha^*)\vec{H}_2^*)\right\}dv+$$

$$-k_0\omega\mu_0\int_{V_2}\left\{\vec{H}_2\cdot(i\omega\mu^*\vec{H}_2^*)+\vec{H}_2^*\cdot(-i\omega\mu\vec{H}_2)\right\}dv+$$

$$k_0\omega\mu_0\int_{V_2}\left\{(\vec{H}_2\cdot\beta^*\vec{E}_2^*)+(\vec{H}_2^*\cdot\beta\vec{E}_2)\right\}dv+$$

$$k_0\omega\mu_0\int_{S_2}(\sigma_s^*+\sigma_s)\left\{(\vec{E}_2\cdot\vec{E}_2^*)-(\vec{E}_2\cdot\vec{n})(\vec{E}_2^*\cdot\vec{n})\right\}da \qquad (2.14)$$

Note that if the permeability, μ, the permittivity, ϵ, and the tensors α and β are scalars times the identity matrix then sufficient conditions for uniqueness are that the real part of σ is positive and the imaginary parts of ϵ and μ are negative and that the real part of σ_s is positive and that the quadratic form associated with the matrix Q defined by

$$Q=$$

$$\begin{pmatrix} A_e & 0 & 0 & Re(\alpha) & 0 & 0 \\ 0 & A_e & 0 & 0 & Re(\alpha) & 0 \\ 0 & 0 & A_e & 0 & 0 & Re(\alpha) \\ Re(\beta) & 0 & 0 & \omega Im(\mu) & 0 & 0 \\ 0 & Re(\beta) & 0 & 0 & \omega Im(\mu) & 0 \\ 0 & 0 & Re(\beta) & 0 & 0 & \omega Im(\mu) \end{pmatrix} \qquad (2.15)$$

where

$$A_e=\omega Im(\epsilon)+Re(\sigma), \qquad (2.16)$$

is positive definite. Thus, in particular, if there is enough domination of the α and β terms by the positive diagonal terms, then this form is positive definite and we do indeed have a unique solution of the electromagnetic interaction problem in a variety of naturally arising function spaces if the impedance sheet conductivity σ_s has a positive real part.

The general uniqueness result is therefore derived by observing that

$$\underset{R\rightarrow\infty}{LIM} \int_{C_R} (\vec{n} \times curl(\vec{E}) - ik_0\vec{E}) \cdot (\vec{n} \times curl(\vec{E}^*) + ik_0\vec{E}^*) da =$$

$$\underset{R\rightarrow\infty}{LIM} \int_{C_R} (| \vec{n} \times curl(\vec{E}) |^2 + k_0^2 | E |^2) da +$$

$$k_0\omega\mu_0 \int_{V_2} \left\{ (\vec{E}_2^* \cdot (i\omega\epsilon + \sigma)\vec{E}_2) + (\vec{E}_2 \cdot (-i\omega\epsilon^* + \sigma^*)\vec{E}_2^*) \right\} dv +$$

$$k_0\omega\mu_0 \int_{V_2} \left\{ (\vec{E}_2^* \cdot (\alpha)\vec{H}_2) + (\vec{E}_2 \cdot (\alpha^*)\vec{H}_2^*) \right\} dv +$$

$$-k_0\omega\mu_0 \int_{V_2} \left\{ (\vec{H}_2 \cdot (i\omega\mu^*\vec{H}_2^*) + (\vec{H}_2^* \cdot (-i\omega\mu\vec{H}_2)) \right\} dv +$$

$$k_0\omega\mu_0 \int_{V_2} \left\{ (\vec{H}_2 \cdot \beta^*\vec{E}_2^*) + (\vec{H}_2^* \cdot \beta\vec{E}_2) \right\} dv +$$

$$k_0\omega\mu_0 \int_{S_2} (\sigma_s^* + \sigma_s) \left\{ (\vec{E}_2 \cdot \vec{E}_2^*) - (\vec{E}_2 \cdot \vec{n})(\vec{E}_2^* \cdot \vec{n}) \right\} da \qquad (2.17)$$

The uniqueness is established by observing that upon taking the limit of all terms as the radius R of C_R becomes infinite that if the difference $\vec{E}$ between two solutions were a nonzero function, then we would get effectively two equations by taking the real and imaginary parts of both sides of the relationship

$$0 = C^2 +$$

$$k_0\omega\mu_0 \int_{V_2} \left\{ (\vec{E}_2^* \cdot (i\omega\epsilon + \sigma)\vec{E}_2) + (\vec{E}_2 \cdot (-i\omega\epsilon^* + \sigma^*)\vec{E}_2^*) \right\} dv +$$

$$k_0\omega\mu_0 \int_{V_2} \left\{ (\vec{E}_2^* \cdot (\alpha)\vec{H}_2) + (\vec{E}_2 \cdot (\alpha^*)\vec{H}_2^*) \right\} dv +$$

$$-k_0\omega\mu_0 \int_{V_2} \left\{ (\vec{H}_2 \cdot (i\omega\mu^*\vec{H}_2^*) + (\vec{H}_2^* \cdot (-i\omega\mu\vec{E}_2)) \right\} dv +$$

$$k_0\omega\mu_0 \int_{V_2} \left\{ (\vec{H}_2 \cdot \beta^*\vec{E}_2^*) + (\vec{H}_2^* \cdot \beta\vec{E}_2) \right\} dv +$$

$$k_0\omega\mu_0 \int_{S_2} (\sigma_s^* + \sigma_s) \left\{ (\vec{E}_2 \cdot \vec{E}_2^*) - (\vec{E}_2 \cdot \vec{n})(\vec{E}_2^* \cdot \vec{n}) \right\} da \qquad (2.18)$$

where C^2 is the real number given by

$$C^2 = \underset{R\rightarrow\infty}{LIM} \int_{C_R} (| \vec{n} \times curl(\vec{E}) |^2 + k_0^2 | E |^2) da \qquad (2.19)$$

Since this is not possible if the electromagnetic parameters are such that the body is dissipative in the sense that the bilinear form acting on the function $(\vec{E}, \vec{H})$ that is defined by

$$b(\vec{E}, \vec{H}) =$$

$$k_0\omega\mu_0 \int_{V_2} \left\{(\vec{E}_2^* \cdot (i\omega\epsilon + \sigma)\vec{E}_2) + (\vec{E}_2 \cdot (-i\omega\epsilon^* + \sigma^*)\vec{E}_2^*)\right\} dv +$$

$$k_0\omega\mu_0 \int_{V_2} \left\{(\vec{E}_2^* \cdot (\alpha)\vec{H}_2) + (\vec{E}_2 \cdot (\alpha^*)\vec{H}_2^*)\right\} dv +$$

$$-k_0\omega\mu_0 \int_{V_2} \left\{(\vec{H}_2 \cdot (i\omega\mu^* \vec{H}_2^*) + (\vec{H}_2^* \cdot (-i\omega\mu\vec{H}_2)\right\} dv +$$

$$k_0\omega\mu_0 \int_{V_2} \left\{(\vec{H}_2 \cdot \beta^* \vec{E}_2^*) + (\vec{H}_2^* \cdot \beta\vec{E}_2)\right\} dv +$$

$$k_0\omega\mu_0 \int_{S_2} (\sigma_s^* + \sigma_s) \left\{(\vec{E}_2 \cdot \vec{E}_2^*) - (\vec{E}_2 \cdot \vec{n})(\vec{E}_2^* \cdot \vec{n})\right\} da \qquad (2.20)$$

is positive definite.

The proof of uniqueness can now be completed in a variety of naturally arising function spaces where the integrals are defined. We see that for an isotropic material that the Silver Mueller radiation conditions, equation (2.3), and our final relation, which is embodied in equations (2.18) and (2.19), imply that if $\vec{E}$ and $\vec{H}$ denote the difference between two solutions of the electromagnetic interaction problem that we have the relationship

$$0 = C^2 + 2k_0\omega\mu_0 \int_\Omega \{(\omega\epsilon'') + Re(\sigma)\} \mid \vec{E} \mid^2 dv$$

$$+2k_0\omega\mu_0 \int_\Omega (\omega\mu'') \mid \vec{H} \mid^2 dv +$$

$$2k_0\omega\mu_0 \int_\Omega Re(\sigma_s) \mid (\vec{E} - (\vec{E} \cdot \vec{n})\vec{n}) \mid^2 da \qquad (2.21)$$

Since we usually write the permeability in the form

$$\mu = \mu' - i\mu'' \qquad (2.22)$$

where μ'' is positive and write the permittivity in the form

$$\epsilon = \epsilon' - i\epsilon'' \qquad (2.23)$$

where ϵ'' is positive, and assume that the real parts of σ and σ_s are not negative, we see that for normal isotropic physical materials, there is only one solution of the electromagnetic interaction problem, but that, while there are many interesting situations in which uniqueness can be established for bianisotropic materials, there is no such simple separate condition on each individual tensor by itself which would guarantee uniqueness of the interaction problem for bianisotropic materials, in the sense that, for isotropic materials, (2.21) tells us immediately that the difference between the electric vectors of the two solutions is identically zero.

3 SEVEN DIMENSIONS

We suggest, here, 7 dimensional theory as a method of solving electromagnetic interaction problems. It is clear that any electromagnetic interaction problem in 7 dimensional space translates into a scattering problem in three dimensional space. Also, problems which are simple in 7 dimensional space often translate into electromagnetic scattering problems in three dimensional space which are seemingly very complex. What remains open is a systematic method of going from problems in 3 dimensional space that we really want to solve into solvable problems in 7 dimensional space. If we consider a seven dimensional vector field

$$\vec{E} = \sum_{j=1}^{7} \{E_j \vec{e}_j\} \tag{3.1}$$

where the components are smooth functions of the spatial variables

$$\boldsymbol{x} = (x_1, x_2, \cdot, \cdot, \cdot, x_7) \tag{3.2}$$

then we have

$$curl(\vec{E}) = \sum_{i=1}^{7} \left[\left(\frac{\partial E_{i+3}}{\partial x_{i+1}} - \frac{\partial E_{i+1}}{\partial x_{i+3}} \right) + \left(\frac{\partial E_{i+6}}{\partial x_{i+2}} - \frac{\partial E_{i+2}}{\partial x_{i+6}} \right) + \left(\frac{\partial E_{i+5}}{\partial x_{i+4}} - \frac{\partial E_{i+4}}{\partial x_{i+5}} \right) \right] \vec{e}_i \tag{3.3}$$

where $\vec{e}_i$ is the unit vector in the direction of the *ith* coordinate axis in 7 dimensional space and

$$E_{i+7} = E_i \tag{3.4}$$

and

$$x_{i+7} = x_i \tag{3.5}$$

for all i in $\{1,2,3,4,5,6,7\}$. The main body of the theory which shows that every vector field in 7 dimensional space is a *curl* plus a *gradient* is the relation

$$curl(curl(\vec{E})) = grad(div(\vec{E})) - \Delta\vec{E} \tag{3.6}$$

This follows from the fact that all nonempty open subsets Ω of $\mathbf{R}^7$ are $\Delta - convex$ in the sense of Hormander ([5], Corollary 3.5.2, page 82) and that consequently if $\vec{F}$ is a vector field in $C^\infty(\Omega, \mathbf{C}^7)$, it follows that there is another vector field $\vec{G}$ such that

$$\Delta\vec{G} = \vec{F} \tag{3.7}$$

Using the previous identity we see that

$$\vec{F} = grad(div(\vec{G})) + curl(curl(-\vec{G})) \tag{3.8}$$

Some work will show that the uniqueness theory also carries over in a natural way to 7 dimensional space without the use of exterior differentials to represent curl operations. The self adjointness of the 3 and 7 dimensional curl on $C_c^\infty(\Omega, \mathbf{C}^n)$ for $n = 3$ and $n = 7$, respectively, permits variational formulations of the problems and the study of weak solutions of interaction problems.

4 SUMMARY

Both complex analysis and embedding of problems in algebras, such as the Cartan algebra mentioned above, have the potential of helping us understand the interaction of electromagnetic waves with new man made materials such as liquid crystals, and the birefringent and piezoelectric materials that have been a source of fascination for centuries. The uniqueness theorem that we proved brings with it the formula for energy density within

a bianisotropic material, including the term involving the product of the electric and magnetic vector; this provides us with a means of checking computer algorithms whose objective is to describe the interaction of radiation with complex materials.

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The need of uniqueness is seen in the next reprint from JDE

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A Characterization of the Linear Partial Differential Operators $P(D)$ which Admit a Nontrivial C^∞ Solution with Support in an Open Prism with Bounded Cross Section

D. K. COHOON

Bell Telephone Laboratories, Inc., Whippany, New Jersey 07981

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Let $P(D)$ denote a linear partial differential operator with constant coefficients in n variables, where presumably $n \geqslant 2$. Let $N^{(1)}, N^{(2)},..., $ and $N^{(n-1)}$ denote a collection of $n-1$ linearly independent vectors in $\mathbb{R}^n$. Any prism with a bounded cross section whose axis of symmetry lies along the direction orthogonal to $N^{(1)}, N^{(2)},..., N^{(n-1)}$ can be embedded in a tube of the form

$$T(N^{(1)}, N^{(2)},..., N^{(n-1)}, R) = \{x \in \mathbb{R}^n : |\langle x, N^{(k)}\rangle| \leqslant R \quad \text{for } k = 1, 2,..., n-1\}.$$

Conversely, any open prism with bounded cross section whose axis of symmetry lies along the direction N orthogonal to $N^{(1)}, N^{(2)},..., N^{(n-1)}$ contains some translate of a tube $T(N^{(1)}, N^{(2)},..., N^{(n-1)}, r)$ for some $r > 0$.

If any factor of $P(D)$ has the property of admitting a C^∞ solution with support in a tube, then $P(D)$ also has this property. Thus, we may without loss of generality assume that $P(D)$ is irreducible in the following sense.

DEFINITION 1. A partial differential operator $P(D)$ is irreducible if $P(D)$ cannot be written as

$$P(D) = P_1(D)\, P_2(D),$$

where both $P_1(D)$ and $P_2(D)$ have degrees which are strictly less than the degree of $P(D)$.

Thus, it is sufficient to characterize the linear partial differential operators $P(D)$ corresponding to which there exists a nontrivial u in $C^\infty(\mathbb{R}^n)$ such that $P(D)\, u(x) = 0$ for all x in $\mathbb{R}^n$, and such that the support of u is contained in $T(N^{(1)},..., N^{(n-1)}, R)$. Thus, let N a nonzero vector which is perpendicular to $N^{(k)}$ for $k = 1, 2,..., n-1$.

Let us transform coordinates according to the rules

$$y_k = \sum_{j=1}^{n} x_j N_j^{(k)} \qquad \text{for} \quad k = 1, 2, \ldots, \quad n-1$$

and

$$y_n = \sum_{j=1}^{n} x_j N_j .$$

Thus, we have that $|\langle x, N^{(k)}\rangle| \leqslant R$ if, and only if, $|y_k| \leqslant R$ for $k = 1, 2, \ldots,$ and $n-1$. Thus, if we assume that the transformed operator, the representation of $P(D)$ in the y coordinates, is $Q(D)$, we ask when there exists a nontrivial $v(y)$ in $C^\infty(\mathbf{R}_y^n)$ such that $Q(D)\,v(y) = 0$ for all y in $\mathbf{R}_y^n$ and such that the support of $v(y)$ is contained in $\{y \in \mathbf{R}_y^n : |y_k| \leqslant R$ for $k = 1, 2, \ldots,$ and $n-1\}$. We give the complete answer to this question with the following theorem.

THEOREM 1. *Suppose $Q(D)$ is an irreducible linear partial differential operator of positive degree. Then there exists a nontrivial $v(y)$ in $C^\infty(\mathbf{R}_y^n)$ such that* (i) *$Q(D)\,v(y) = 0$ for all y in $\mathbf{R}_y^n$ and* (ii) *such that the support of $v(y)$ is contained in $\{y \in \mathbf{R}_y^n : |y_k| \leqslant R$ for $k = 1, 2, \ldots,$ and $n-1\}$ if, and only if $Q(D)$ is up to multiplication by a nonzero scalar of the form*

$$Q(D) = D_n^{\,m} + \sum_{|\alpha| < m} a_\alpha D^\alpha. \tag{3}$$

Proof. First, suppose that $Q(D)$ has the form (3). By Theorem 3.11 of Treves [3], there is a function $U(\zeta', x_n)$ which is an entire function of ζ' in $\mathbf{C}^{n-1}$ and x_n which satisfies

$$Q(\zeta', D_n)\, U(\zeta', y_n) = 0, \tag{4}$$

$$D_n^k U(\zeta', 0) = 0 \qquad \text{for} \quad k = 0, 1, \ldots, m-2, \tag{5}$$

$$D_n^{m-1} U(\zeta', 0) = 1, \tag{6}$$

and

$$|U(\zeta', y_n)| \leqslant \frac{|y_n|^{m-1}}{(m-1)!} \exp\Big(\sum_{k=1}^{m} |\sigma_k(\zeta')|^{1/k}\, |y_n|\Big) \tag{7}$$

for all ζ' in $\mathbf{C}^{n-1}$ and all y_n in $\mathbf{R}$, where

$$Q(\zeta', D_n) = \sum_{k=1}^{m} \sigma_k(\zeta')\, D_n^{m-k} + D_n^{\,m}. \tag{8}$$

For all $\zeta' = (\zeta_1, \zeta_2, ..., \zeta_{n-1})$ in $\mathbf{C}^{n-1}$, let us define

$$\operatorname{Re}\zeta' = (\operatorname{Re}\zeta_1, \operatorname{Re}\zeta_2, ..., \operatorname{Re}\zeta_{n-1}), \tag{9}$$

$$\operatorname{Im}\zeta' = (\operatorname{Im}\zeta_1, \operatorname{Im}\zeta_2, ..., \operatorname{Im}\zeta_{n-1}), \tag{10}$$

and

$$|\zeta'|_q = (|\zeta_1|^q + |\zeta_2|^q + \cdots + |\zeta_{n-1}|^q)^{1/q} \tag{11}$$

for all numbers $q \geqslant 1$. Notice that our hypothesis tell us that the degree of $\sigma_k(\zeta')$ regarded as a polynomial in $\zeta_1, \zeta_2, ...,$ and ζ_{n-1} is necessarily of degree $k - 1$ or smaller. But this means that there is a constant $B > 0$ depending only on $Q(D)$ such that

$$\sum_{k=1}^{m} |\sigma_k(\zeta')|^{1/k} \leqslant B(1 + |\zeta'|_1)^{(m-1)/m}. \tag{12}$$

We have also that for any $B_1 \geqslant n^b$ and any $B_1' \leqslant 1/n$, where n is a positive integer and b is a positive number, that

$$B_1'\left(\sum_{k=1}^{n-1} |\zeta_k|^b\right) \leqslant \left(\sum_{k=1}^{n-1} |\zeta_k|\right)^b \tag{13}$$

and

$$(1 + |\zeta'|_1)^b \leqslant B_1\left(1 + \sum_{k=1}^{n-1} |\zeta_k|^b\right).$$

We will suppose that ϕ is a member of $C^\infty(\mathbf{R}^{n-1})$ of class $\gamma^{(\delta)}(\mathbf{R}^{n-1})$ (see Hormander [2], p. 146) whose support is contained in

$$\{(y_1, ..., y_{n-1}) \in \mathbf{R}_y^{n-1} : |y_k| \leqslant R/2 \quad \text{for } k = 1, 2, ..., \text{ and } n - 1\}.$$

From the Paley-Wiener Theorem, we deduce that for every $C > 0$ there is a $K_C > 0$ such that

$$|\hat{\phi}(\zeta')| \leqslant K_C \exp\left((R/2)\,|\operatorname{Im}\zeta'|_1 - C\sum_{k=1}^{n-1} |\operatorname{Re}\zeta_k|^{1/\delta}\right). \tag{14}$$

Using (13) with $b = 1/\delta$, we deduce that

$$-C\sum_{k=1}^{n-1} |\operatorname{Re}\zeta_k|^{1/\delta} \leqslant -\left(\frac{C}{B_1}\right)\left(\sum_{k=1}^{n-1} |\operatorname{Re}\zeta_k|\right)^{1/\delta} + C. \tag{15}$$

Substituting (15) into (14), we deduce that for every $C > 0$ there exists a constant $K_C' = \exp(B_1 C)\, K_{(B_1 C)}$ such that

$$|\hat{\phi}(\zeta')| \leqslant K_C' \exp((R/2)\,|\operatorname{Im}\zeta'|_1 - C\,|\operatorname{Re}\zeta'|_1^{1/\delta}). \tag{16}$$

We introduce the constant $K''_C = K'_{(C/B_1')}$. We deduce that, conversely, if (16) is satisfied, then from (13) it follows that (14) is satisfied with K_C replaced by K''_C. Thus, (16) is a necessary and sufficient condition that $\hat{\phi}(\zeta')$ be the Fourier transform of a C^∞ function of Gevry class δ with support in $\{y \in \mathbf{R}^{n-1} : |y_k| \leqslant R/2$ for $k = 1, 2, \ldots,$ and $n-1\}$. Now we consider $\hat{u}(\zeta', y_n) = U(\zeta', y_n)\hat{\phi}(\zeta')$. We have that

$$Q(\zeta', D_n)\, \hat{u}(\zeta', y_n) = 0, \tag{17}$$

$$D_n^k \hat{u}(\zeta', 0) = 0, \quad K = 0, 1, \ldots, m-2, \tag{18}$$

$$D_n^{m-1} \hat{u}(\zeta', 0) = \hat{\phi}(\zeta'). \tag{19}$$

Combining (7) and (13) we deduce that for some $B_2 > 0$

$$| U(\zeta', y_n)| \leqslant \frac{B_2 \,| y_n |^{m-1}}{(m-1)!} \times \exp(| y_n | B_2 | \operatorname{Re} \zeta' |_1^{(m-1)/m} + | y_n | B_2 | \operatorname{Im} \zeta' |_1^{(m-1)/m}).$$

There is a constant $K'''_C = K'_{2C}(B_2/(m-1)!)$ such that

$$| \hat{u}(\zeta, y_n)| \leqslant K'''_C \,| y_n |^{m-1} f(\zeta', y_n)\, g(\zeta', y_n), \tag{20}$$

where

$$\exp(B_2 | y_n | | \operatorname{Re} \zeta' |_1^{(m-1)/m} - 2C | \operatorname{Re} \zeta' |_1^{(1/\delta)}) = f(\zeta', x_n) \tag{21}$$

and

$$\exp(| y_n | B_2 | \operatorname{Im} \zeta' |_1^{(m-1)/m} + (R/2) | \operatorname{Im} \zeta' |_1) = g(\zeta', x_n). \tag{22}$$

Choose δ so that $1 > 1/\delta > (m-1)/m$. Then there is a $B_3 > 0$ depending on y_n such that

$$\exp(B_2 | y_n | | \operatorname{Re} \zeta' |_1^{(m-1)/m} - C | \operatorname{Re} \zeta' |_1^{(1/\delta)}) \leqslant B_3 \tag{23}$$

for all ζ' in $\mathbf{C}^{n-1}$. Also, there is a constant $B_4 > 0$ depending on y_n such that

$$\exp(| y_n | B_2 | \operatorname{Im} \zeta' |_1^{(m-1)/m} - (R/2) | \operatorname{Im} \zeta' |_1) \leqslant B_4 .$$

Thus, we conclude that for every $C > 0$ there is a $K'''_C > 0$ and a constant B_5 depending on y_n such that

$$| \hat{u}(\zeta', y_n)| \leqslant K'''_C B_5 \exp(R | \operatorname{Im} \zeta' |_1 - C | \operatorname{Re} \zeta' |_1^{(1/\delta)}) \tag{24}$$

for all ζ' in $\mathbf{C}^{n-1}$. Using (24) and the Paley-Weiner Theorem (Lemma 5.7.2 of Hörmander [1]), it follows that $\hat{u}(\zeta', y_n)$ is for each y_n the Fourier transform

of an element $v_{y_n}(y_1, \ldots, y_{n-1}) = u(y_1, \ldots, y_{n-1}, y_n)$ of $C^\infty(\mathbf{R}^{n-1})$ whose support is contained in $\{(y_1, \ldots, y_{n-1}) \in \mathbf{R}^{n-1} : |y_k| \leqslant R$ for $k = 1, 2, \ldots,$ and $n-1\}$. For we use the fact that $U(\zeta', y_n)$ is an entire function of ζ' and y_n and the Cauchy integral theorem to deduce that

$$\left| \frac{\partial^k U(\zeta', y_n)}{\partial {y_n}^k} \right| \leqslant k! \, \frac{(|y_n| + 1)^{m-1}}{(m-1)!} \exp(B \, | y_n | (1 + |\zeta'|_1)^{(m-1)/m}). \quad (25)$$

Repeating the argument using (25), we easily deduce that all derivatives of $\hat{u}(\zeta', y_n)$ with respect to y_n satisfy an inequality of the form (24). This completes the proof that if $Q(D)$ is of the form (3) then (i) and (ii) are satisfied.

Now we want to show that if (i) and (ii) are satisfied, then $Q(D)$ is necessarily of the form (3).

LEMMA 1. *Let $Q(D)$ be an irreducible linear partial differential operator of degree $m > 0$ with constant coefficients. Suppose that there is a nontrivial function u in $C^\infty(\mathbf{R}^n)$ satisfying $Q(D)\, u(y) = 0$ for all y in $\mathbf{R}^n$ and $u(y) = 0$ if $|y_k| \geqslant R$ for $k = 1, 2, \ldots,$ or $n-1$. Then $Q_m(D)$ is hyperbolic in the direction N for every N in $\mathbf{R}^n$ which is not orthogonal to* $(0, \ldots, 0, 1)$.

Proof. We consider the plane $\Sigma(N) = \{x \in \mathbf{R}^n : \langle x, N\rangle = 0\}$, where N is a vector in $\mathbf{R}^n$ that is not orthogonal to $(0, \ldots, 0, 1)$. Let

$$B = \{x \in \mathbf{R}^n : |x_k| \leqslant R \text{ for } k = 1, 2, \ldots, n-1\}.$$

Since $N_n \neq 0$, it is obvious that $\Sigma(N) \cap B$ is compact. We now need only an application of Theorem 5.7.2 of Hörmander [2] to deduce that unless N is a hyperbolic direction, u vanishes identically in $\Sigma(N)$, and by translations we deduce that u vansishes identically, which contradicts the hypothesis of the lemma. This completes the proof of Lemma 1.

LEMMA 2. *Suppose $Q_m(D)$ is a homogeneous differential operator of degree m which is hyperbolic in the direction $N = (N_1, \ldots, N_n)$ whenever $N_n \neq 0$. Then there does not exist any $(\xi_1, \ldots, \xi_{n-1})$ in $\mathbf{R}^{n-1}$ such that for some nonzero complex number ζ, we have $Q_m(\xi_1 + \zeta N_1, \ldots, \xi_{n-1} + \zeta N_{n-1}, \zeta N_n) = 0$ where $(N_1, \ldots, N_{n-1}, N_n) \in \mathbf{R}^n$ and $N_n \neq 0$.*

Suppose that for some $(\xi_1, \ldots, \xi_{n-1}) \in \mathbf{R}^{n-1}$ we could find a nonzero complex number ζ such that

$$Q_m(\xi_1 + \zeta N_1, \xi_2 + \zeta N_2, \ldots, \xi_{n-1} + \zeta N_{n-1}, \zeta N_n) = 0. \quad (26)$$

But Theorem 5.3.3 of Hörmander [2] tells us that ζ must have been real. But then we conclude from the hypothesis of the lemma that

$$(\xi_1 + \zeta N_1, \xi_2 + \zeta N_2, \ldots, \xi_{n-1} + \zeta N_{n-1}, \zeta N_n)$$

is a hyperbolic direction of $Q_m(D)$ and, consequently, that

$$Q_m(\xi_1 + \zeta N_1, \xi_2 + \zeta N_2, \dots, \xi_{n-1} + \zeta N_{n-1}, \zeta N_n) \neq 0. \tag{27}$$

This is a contradiction, and the proof of Lemma 2 is complete.

We suppose hypothesis (i) and (ii) are satisfied. Since $(0,\dots, 0, 1)$ is a noncharacteristic direction of $Q_m(D)$, it follows that

$$Q_m(\xi_1, \dots, \xi_{n-1}, \zeta) = \sum_{k=0}^{m-1} \left(\sum_{|\beta|=m-k} a_{(\beta,k)} \xi_1^{\beta_1} \cdots \xi_{n-1}^{\beta_{n-1}} \right) \zeta^k + b\zeta^m,$$

where $b \neq 0$. Since Lemma 1 and Lemma 2 tell us that $Q_m(\xi_1, \dots, \xi_{n-1}, \zeta)$, regarded as a polynomial in ζ, has only $\zeta = 0$ as a root for all $(\xi_1, \dots, \xi_{n-1})$ in $\mathbf{R}^{n-1}$, we conclude that $a_{(\beta,k)} = 0$ for all β in $\mathbf{N}^{n-1}$ with $|\beta| = m - k$ for all k in $\{0, 1,\dots, m-1\}$. Thus, $Q_m(D) = bD_n^m$. This completes the proof of Theorem 1.

The author was able to prove the following result about systems admitting a nontrivial vector valued C^∞ solution with support in an open prism with bounded cross section.

THEOREM 2. *Let L be a $p \times p$ matrix, each entry of which is a linear partial differential operator with constant coefficients. Suppose that* $\det(L)$ *is a partial differential operator of positive degree. Then* (i) *there is a nontrivial $\bar{u}$ in* Ker(L) ∩ $C^\infty(\mathbf{R}^n, \mathbf{C}^p)$, *and $\bar{u}$ vansishes outside of an open prism with bounded cross section if, and only if,* (ii) *there is a nontrivial v in $C^\infty(\mathbf{R}^n)$ such that* $\det(L)\, v(x) = 0$ *for all x in $\mathbf{R}^n$ and such that v vansishes outside the same open prism with bounded cross section.*

Proof. That (i) implies (ii) is trivial. Now suppose (ii) holds. In light of Theorem 1 there is no loss of generality in assuming that $T = \det(L)$ is of the form (3).

Let $\Gamma(\hat{L})$ and $\hat{T}$ denote the Fourier transforms with respect to $x_1, x_2, \dots,$ and x_{n-1} of $\Gamma(L)$ and T, respectively. Let $Y(x_n)\, U(\zeta', x_n)$ denote the fundamental solution of $\hat{T}$ described by Theorem 3.11 of Treves [3]. We can use techniques similar to those used in proving Theorem 2 of Cohoon [1] to show that the (j, k)-th entry of $\Gamma(\hat{L})$ fails to annihilate $U(\zeta', x_n)$ for some j and k in $\{1, 2,\dots, p\}$. Thus, if $\hat{\phi}(\zeta')$ is an entire function satisfying (16), the function $\bar{u} = \Gamma(L)\, \bar{v}$ satisfies condition (i) of Theorem 2, where $v_r = 0$ for $r \neq k$, and v_k is the inverse Fourier transform with respect to ζ' of $\phi(\zeta')\, U(\zeta', x_n)$. This completes the proof of Theorem 2.

In this theorem, I consider $p \times p$ matrices, L, each entry of which is a linear partial differential operator with constant coefficients in n independent variables. In addition, I assume that $\det L$ was a nontrivial partial differential

operator of positive degree. I remark that if $\det(L) = C$, a nonzero complex number, then the equation

$$L\phi = 0$$

has no nontrivial solutions in $C^\infty(\mathbf{R}^n, \mathbf{C}^p)$. On the other hand, it can be shown by induction on p that if $\det L = 0$, the equation $L\phi = 0$ has a nontrivial solution ϕ in $C_0^\infty(\mathbf{R}^n, \mathbf{C}^p)$ with support in an arbitrarily small neighborhood of 0.

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EXACT FORMULAS FOR REACTIVE INTEGRALS ARISING IN THE ELECTROMAGNETIC SCATTERING PROBLEM FOR NONHOMOGENEOUS, ANISOTROPIC BODIES OF REVOLUTION - - Evaluation of Integrals of Functions Defined by Riemann Surfaces and zero finding with applications of Electromagnetic Theory to the treatment of cancer

D. K. Cohoon

March 2, 1992

Contents

Bodies of revolution are structures, including aerosol particles which are fibers and flakes, or larger bodies of electromagnetic material which have an axis of symmetry with the property that if an electromagnetic wave interrogates the scattering body before and after any partial rotation about this axis, this impinging electromagnetic wave can see no difference. The electromagnetic interaction problem is complicated by the fact that every portion of this body of revolution as it is stimulated by the impinging radiation communicates with every other portion of the body of revolution. Because of the rotational symmetry, it seems prudent to represent the components of the induced electric and magnetic fields as a Fourier series and solve an integral equation formulation of the scattering problem by solving for Fourier components of piecewise polynomial approximations of the field components within each cell of the body. This Fourier analysis involves trigonometric integrals which when transformed to the complex plane would involve analysis of functions defined on a Riemann surface. We provide in this paper a new way of evaluating these integrals using only information around an essential singularity.

Bodies of revolution also include bodies that have spheres, cylinders, oblate or prolate spheroids, or a torus as boundaries of a material that responds to the radiation. Analyzing the latter may have some benefit in the controlled thermonuclear fusion problem ([1]) as we could then by computer analysis design a material with ultra high absorption efficiency. The low cost of computer experimentation may also permit one to design an ultraviolet light absorbing aerosol that will protect man and animals from the high levels of ultra violet B radiation that they may soon be experiencing as a result of the ozone depletion. The material within a body of revolution may have tensor properties, but the body, together with its properties is still unchanged by any partial rotation about the axis of symmetry. This could include, for example, a tensor material which has one property in the direction of the axis of rotation and another property in all directions going radially outward from this axis of symmetry. Here, one might think of cutting a sphere out of a cylinder comprised of closely packed dielectric needles. Externally using visible light this sphere might look to our eye like any other round ball, but to electromagnetic waves polarized in the direction of the axis (and consequently parallel to these soft dielectric needles) and to those electromagnetic waves polarized in a direction perpendicular to the axis of revolution, the two responses would be completely different. Materials such as these are used in current liquid crystal devices, and the full development of the theory may in the future result in a more healthful replacement for the present radiating video displays. The details of the connection between the integrals discussed in this paper and electromagnetic interaction problems are found in many sources ([21]) but is lucidly explained in Glisson and Wilton ([13]).

We also in this paper describe a method of finding zeros of analytic functions using homotopy methods. This method has a variety of design applications, including those for optical computers.

In addition we introduce a type of three dimensional complex analysis that permits one at low cost to develop a realistic model of the man in an electromagnetic field. This will make possible the design of a safe and effective system for killing localized cancer tumors with microwaves by raising the temperature of the tumor by 4 degrees Centigrade without raising the temperature of the normal tissue to this level.

1 Exact Evaluation of Integrals

We introduce a function ξ which represents the distance between two points, represented in cylindrical coordinates as (ρ, θ, z) and $(\tilde{\rho}, \tilde{\theta}, \tilde{z})$ so that

$$\xi = \sqrt{\rho^2 + \tilde{\rho}^2 + z^2 + \tilde{z}^2 - 2 \cdot \rho \cdot \tilde{\rho} \cdot cos(\psi)} \tag{1.1}$$

where ψ is the difference between θ and $\tilde{\theta}$. The integral under consideration is

$$I^{(c)}_{(\ell,m)} = \int_0^{2\pi} \frac{exp(i\xi)}{\xi^\ell} \cdot cos(m\psi)d\psi \tag{1.2}$$

where

$$\xi^2 = (A - 2Bcos(\psi)) \tag{1.3}$$

where

$$A^2 > 4B^2 \tag{1.4}$$

where A is positive and m is a nonnegative integer. The function ξ defined by equation (1.3) is an algebraic function defined by a Riemann surface if you make the normal extension to the complex plane by rewriting equation (1.3) in the form,

$$\xi^2 = (A - B \cdot (\zeta + 1/\zeta) \tag{1.5}$$

where if ζ is equal to $exp(i\psi)$, then

$$2 \cdot cos(\psi) = (\zeta + 1/\zeta) \tag{1.6}$$

The rational function ξ^2 has a simple pole at the origin and one zero inside the unit circle and another zero outside the unit circle. The algebraic function ξ is defined by a Riemann surface with a Branch cut from the orgin to a zero,

$$\zeta_1 = \frac{A - \sqrt{A^2 - 4B^2}}{2 \cdot B} = \frac{2B}{A + (A^2 - 4B^2} \tag{1.7}$$

of the function ξ^2 that is inside the unit circle

$$|\zeta| = 1 \tag{1.8}$$

and a branch cut from ∞ to the zero,

$$\zeta_2 = \frac{A + \sqrt{A^2 - 4B^2}}{2 \cdot B} \tag{1.9}$$

that is outside the unit circle. If we use the argument function defined by

$$Arg(x + iy) = \theta \tag{1.10}$$

where if

$$r = \sqrt{x^2 + y^2} \tag{1.11}$$

then, θ, the value of the argument function defined by equation (1.10) is such that

$$r\cos(\theta) + ir\sin(\theta) = x + iy \quad 0 \geq \theta < 2\pi \tag{1.12}$$

We can use a Riemann surface to define the square root of the meromorphic function ξ^2 defined by equation (1.9) to define the algebraic function ξ or use the argument function Arg defined by equations (1.10) and (1.12)

$$\xi =$$

$$i\left[\frac{B \mid \zeta - \zeta_1 \mid\mid \zeta - \zeta_2 \mid}{\mid \zeta \mid}\right]^{1/2} exp(i(1/2)(Arg(\zeta - \zeta_1) + Arg(\zeta - \zeta_2) - Arg(\zeta)) \tag{1.13}$$

Thus, an integral of a holomorphic function of ξ around the unit circle will be equal to the integral of the same function around a rectangle inside the unit circle which contains the slit from the origin to ζ_1. We show how information around the essential singularity will give us an exact formula; our formula will be checked by direct Fourier analysis observing that

$$\frac{1}{(A - 2 \cdot B \cdot cos(\psi))^{1/2}} = \left(\frac{1}{\sqrt{A}}\right) \sum_{k=0}^{\infty} P_k^{(1)} \cdot 2^k \cdot cos^k(\psi) \tag{1.14}$$

where

$$P_k^{(1)} = \left(\frac{B}{A}\right)^k \left[\prod_{j=1}^{k} \left(\frac{2 \cdot j - 1}{2 \cdot j}\right)\right] \tag{1.15}$$

which means that we can think in terms of representing powers of $cos(\psi)$ as a Fourier series.

2 Reactive Integrals

An exact formula for the values of the reactive integrals has been obtained, and furthermore, the cost of finding the value of the reactive integrals, which were in all other works (e.g. [21] and [13]) carried out by a numerical integration scheme whose computation time increases directly with the m appearing in equation (1.2), is with this exact formula independent of m. Furthermore, this exact formula depends only on values at the essential singularity expansion at ζ equals 0. These formulas have been validated by Fourier expansion and by numerical comparison to 12 or more decimal places with the straightforward numerical integration scheme described in the previous section. The first essential singularity expansion has the form,

$$\frac{1}{\xi^{\ell}} = \frac{1}{A^{\ell/2}}\left[\sum_{q=0}^{\infty} P_q^{(\ell)} \left(\zeta + \frac{1}{\zeta}\right)^q\right] \tag{2.1}$$

We can expand the function $cos(\xi)$ by the series

$$cos(\xi) = \sum_{k=0}^{\infty}\left[\frac{(-1)^k \xi^{2k}}{(2k)!}\right] = \sum_{s=0}^{\infty} C_s \left(\zeta + \frac{1}{\zeta}\right)^s \tag{2.2}$$

We now use the Cauchy product and equations (2.1) and (2.2) to write

$$\left\{\frac{cos(\xi)}{\xi^{\ell}}\right\} = \frac{1}{A^{\ell/2}}\left[\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} C_{k-j}P_k^{(\ell)}\right)\left(\zeta+\frac{1}{\zeta}\right)^j\right] \tag{2.3}$$

If we define

$$D_j^{(\ell)} = \sum_{k=0}^{j} C_{k-j}P_k^{(\ell)} \tag{2.4}$$

then equation (2.3) implies that

$$\left\{\frac{cos(\xi)}{\xi^{\ell}}\right\} = \frac{1}{A^{\ell/2}}\left[\sum_{j=0}^{\infty} D_j^{(\ell)}\left(\zeta+\frac{1}{\zeta}\right)^j\right] \tag{2.5}$$

There are two expansions of even and odd powers of $cos(\psi)$ which enable us to evaluate these contour integrals. The even powers of cos are given ([17], p 24-26) by

$$cos^{2\cdot\ell}(\psi) = \frac{1}{2^{2\cdot\ell}}\left[\sum_{k=0}^{\ell-1}\left\{2\begin{pmatrix}2\cdot\ell\\k\end{pmatrix}\cdot cos(2\cdot(\ell-k)\psi)\right\} + \begin{pmatrix}2\cdot\ell\\\ell\end{pmatrix}\right] \tag{2.6}$$

and the Fourier expansion of an odd power of the cosine ([17], pp 24-26) is

$$cos^{2\cdot q-1}(\psi) = \frac{1}{2^{2\cdot q-1}}\left[\sum_{k=0}^{q-1}\left\{2\cdot\begin{pmatrix}2\cdot q-1\\k\end{pmatrix}\cdot cos((2\cdot q-2\cdot k-1)\psi)\right\}\right] \tag{2.7}$$

If we assume that ζ is equal to $exp(i\psi)$, then we can use equations (2.6) and (2.7) and the relationship

$$\int_{-\pi}^{\pi}\left(\zeta+\frac{1}{\zeta}\right)^j cos(m\psi)d\psi = \int_{-\pi}^{\pi}\left(2^j\cdot cos^j(\psi)cos(m\psi)\right)d\psi \tag{2.8}$$

to evaluate the reactive integrals. We consider first the case where j is equal to $2\cdot\ell$ and use equation (2.8) and equation (2.6) to obtain for positive even integers m not exceeding $2\cdot\ell$ the relationship,

$$\int_{-\pi}^{\pi}\left(\zeta+\frac{1}{\zeta}\right)^{2\cdot\ell}\left(\frac{1}{2}\right)\left(\zeta^m+\frac{1}{\zeta^m}\right)\frac{d\zeta}{i\zeta} = 2\pi\begin{pmatrix}2\cdot\ell\\\ell-m/2\end{pmatrix} \tag{2.9}$$

to evaluate the reactive integrals. In the case where j is equal to $2\cdot\ell$ and

$$2\ell - 2k \tag{2.10}$$

we observe that

$$k = \frac{j-m}{2} \tag{2.11}$$

In considering the case where j is equal to $2 \cdot q - 1$ we use the fact that in the case where m is an odd integer and j is equal to $2 \cdot q - 1$ that

$$\int_{-\pi}^{\pi} \left(\zeta + \frac{1}{\zeta}\right)^{2\cdot q-1} \left(\frac{1}{2}\right) \left(\zeta^m + \frac{1}{\zeta^m}\right) \frac{d\zeta}{i\zeta} = 2\pi \binom{2 \cdot q - 1}{(2 \cdot q - 1 - m)/2} \tag{2.12}$$

We conclude that equation (2.5) implies that

$$\int_{-\pi}^{\pi} \left\{\frac{cos(\xi)}{\xi^{\ell}}\right\} \cdot cos(m\psi) d\psi =$$

$$\frac{1}{A^{\ell/2}} \left[\sum_{j=0}^{\infty} D^{(\ell)}_{m+2\cdot j} \cdot 2\pi \cdot \binom{m + 2 \cdot j}{j}\right] \tag{2.13}$$

We next develop an expression for integrals involving $sin(\xi)/\xi$ by first observing that

$$\frac{sin(\xi)}{\xi} = \sum_{s=0}^{\infty} S_s \left(\zeta + \frac{1}{\zeta}\right)^{s} \tag{2.14}$$

We then make use of the fact that

$$\frac{sin(\xi)}{\xi^{\ell}} = \left(\frac{1}{\xi^{\ell-1}}\right) \left(\frac{sin(\xi)}{\xi}\right) =$$

$$\frac{1}{A^{(\ell-1)/2}} \left[\sum_{q=0}^{\infty} P_q^{(\ell-1)} \left(\zeta + \frac{1}{\zeta}\right)^{q} \frac{sin(\xi)}{\xi}\right] \tag{2.15}$$

Multiplying the series given by equations (2.14) and (2.15) we see that

$$\frac{sin(\xi)}{\xi^{\ell}} = \frac{1}{A^{(\ell-1)/2}} \left[\sum_{j=0}^{\infty} E_j^{(\ell)} \left(\zeta + \frac{1}{\zeta}\right)^{j}\right] \tag{2.16}$$

where

$$E_j^{(\ell)} = \sum_{k=0}^{j} \left(S_{k-j} \cdot P_k^{(\ell-1)}\right) \tag{2.17}$$

Thus, we conclude that

$$\int_{-\pi}^{\pi} \left(\frac{sin(\xi)}{\xi^{\ell}}\right) cos(m\psi) d\psi =$$

$$\frac{1}{A^{(\ell-1)/2}} \left[\sum_{q=0}^{\infty} \left\{E^{(\ell)}_{m+2\cdot q} 2\pi\right\} \binom{m + 2 \cdot q}{q}\right] \tag{2.18}$$

By making use of the identity

$$\binom{j}{(j - m)/2} = \binom{j}{(j + m)/2} \tag{2.19}$$

the formula (2.18) and the formula (2.13) can be given a different look, but several different numerical checks all agreed to machine precision. These formulas were checked by numerical computation using Gaussian quadrature. In the case where the observation point is close to the variable of integration or said differently when $2B$ is very nearly as large as A, then the series can converge slowly, but they can still be evaluated accurately if one uses Euler's method of accelerating convergence of sums ([14], pp 201 - 207). The following table shows a weakness in the method without the use of accelerated convergence. When ρ and $\tilde{\rho}$ are both equal to 1 and when z and $\tilde{z}$ are both equal to 1.1 as in equation (1.1) and we just use 139 terms for the geometric series and we make use of the fact that $cos(\xi)$ divided by ξ^2 is meromorphic and use the contribution to the reactive integral from the simple pole at the zero $zeta_1$ inside the unit circle given by equation (1.7) versus using the Riemann surface concept with just a small number of terms

Essential Singularity Contribution 139 terms	*Gaussian Quadrature Integration*	*Pole and* $\zeta = 0$ *Contribution*	*Mode Index*
3.724	3.726	3.726	1
2.842	2.844	2.844	2
2.110	2.112	2.112	3
.472	.4739	.4739	8

For the difficult cases described in the above table over 6000 thousand terms were used along with accelerated convergence and 15 decimal place agreement between the three methods was achieved. The following table shows the capabilities of the formulae when augmented by Euler's method for accelerated convergence for the case where ρ is equal to 1, $\tilde{\rho}$ is equal to $1+0.2$, z is equal to 1, and $\tilde{z}$ is equal to $1+0.2$. Using the Riemann surface concept and carrying out an expansion about the essential singularity we have

$\frac{exp(i\xi)}{\xi^n}$ *SINGULARITY EXPANSION METHOD*	$cos(m\psi)$ *ENHANCED GAUSSIAN QUADRATURE*	$d\psi$ $exp(im\psi)$ *MODE INDEX*
$3.75346548 - (1.5112968)i$	$3.75346548 - (1.5112968)i$	1
$.382950948 - (.0520636761)i$	$.382950948 - (.0520636761)i$	10
$.0291669710 - (.00235093482)i$	$.0291669710 - (.00235093483)i$	20

Also, the terms of the expansions of $sin(\xi)$ and $cos(\xi)$ can be determined by exact formulas by making use of the Jensen Voller's formula, a variant of the Faa Di Bruno formula ([9]). For example the term C_0 appearing in equation (2.2) is given by

$$C_0 \; = \; cos(\sqrt{A}) \tag{2.20}$$

An alternative representation of these integrals in terms of known special functions is found in a much more general setting in Chapter 7 of (Carlson, [5]), where the integral

$$I(r,m) \; = \; \int_0^{2\cdot\pi} (A \; - 2 \cdot Bcos(\psi))^{r/2} \cdot cos(m\psi) d\psi \tag{2.21}$$

arises as a special case, and Carlson's condition for rapid ordinary convergence of the series which states that the ratio

$$\mathcal{R} = \frac{A - 2 \cdot B}{A + 2 \cdot B} \tag{2.22}$$

stay away from zero is equivalent to ours.

3 Riemann Surfaces and Electromagnetic Pulses

In this section we treat the interaction of electromagnetic pulses with Lorentz media ([4]) and Debye Media layered materials exactly using a Riemann surface to define the complex propagation constant of the dispersive material as a function of complex frequency. We show in an elementary way how one could with nonmagnetic human tissue estimate how harmful, in terms of X ray or Gamma ray component contribution, it would be for humans to be exposed to radar pulses or laser pulses.

3.1 Lorentz Medium

The propagation constant for a Lorentz medium is defined by the square root of

$$k^2 = k_0^2 \left(1 + \frac{a_j^2}{-\omega^2 + i\omega g_j + \omega_j^2}\right) \tag{3.1}$$

where

$$k_0 = \frac{\omega}{c} \tag{3.2}$$

with c being the vacuum speed of light. If we follow Brillouin ([4]) and introduce the variables,

$$\rho_j = \frac{g_j}{2} \tag{3.3}$$

$$\omega_\infty^+ = \sqrt{\omega_j^2 - \rho_j^2} + i\rho_j, \tag{3.4}$$

$$\omega_\infty^- = -\sqrt{\omega_j^2 - \rho_j^2} + i\rho_j, \tag{3.5}$$

$$\omega_0^+ = \sqrt{\omega_j^2 + a_j^2 - \rho_j^2} + i\rho_j, \tag{3.6}$$

and

$$\omega_0^- = -\sqrt{\omega_j^2 + a_j^2 - \rho_j^2} + i\rho_j \tag{3.7}$$

We define branch cuts in the upper half plane from ω_0^- to ω_∞^- in the second quadrant and from ω_∞^+ to ω_0^+ in quadrant I of the upper half plane. The algebraic function k/k_0, where

$$k_0 = \frac{\omega}{c} \tag{3.8}$$

is defined in terms of the argument function

$$Arg(x + iy) = \theta \tag{3.9}$$

where θ is between 0 and $2 \cdot \pi$ and

$$(x, y) = \sqrt{x^2 + y^2}(cos(\theta), sin(\theta)) \tag{3.10}$$

by the relation

$$(k/k_0)(\omega) = \left[\frac{|(\omega - \omega_0^+)||(\omega - \omega_0^-)|}{|(\omega - \omega_\infty^+)||(\omega - \omega_\infty^-)|}\right]^{1/2} \cdot \left[\frac{exp(i\left\{Arg(\omega - \omega_0^+) + Arg(\omega - \omega_0^-)\right\}/2)}{exp(i\left\{Arg(\omega - \omega_\infty^+) + Arg(\omega - \omega_\infty^-)\right\}/2)}\right] \tag{3.11}$$

If we have a sinusoidal electromagnetic pulse with central frequency ω_0 of duration T seconds with field strength E_0 volts per meter, then the value of the electric vector of the incoming electromagnetic radiation is has a Fourier transform that is equal to

$$\mathcal{F}(E^i) = E_0\left[\frac{\omega_0(1 - exp(-i\omega T))}{\omega_0^2 - \omega^2}\right] \tag{3.12}$$

We can also define, in terms of equation (3.12), the coefficients defining the Fourier transform of the electromagnetic pulse returned from the half space and the electromagnetic pulse transmitted into the half space by the rule,

$$C_s(\omega) = \mathcal{F}(E^i)\left(\frac{k_0(\omega) - k(\omega)}{k_0(\omega) + k(\omega)}\right) \tag{3.13}$$

and

$$C_t(\omega) = \mathcal{F}(E^i)\left(\frac{2 \cdot k_0(\omega)}{k_0(\omega) + k(\omega)}\right) \tag{3.14}$$

so that the Fourier transformed electric field vector component inside the half space is

$$\mathcal{F}(E^t)(\omega, x) = \mathcal{F}(E^i) \cdot C_t(\omega) \cdot exp(-ik(\omega)x) \tag{3.15}$$

where x is the distance into the half space. The conductivity times the square of the absolute value of $\mathcal{F}(E^t)$ defined by equation (3.15) would give us the energy density at frequency ω at a depth x into the nonmagnetic half space.

3.2 Debye Medium

A slight modification of the analysis of the previous section will permit one the carry out the analysis for a Debye medium(Daniel, [11]), an appropriate model for human tissue exposed to radar pulses. We define the propagation constant k in terms of its square

$$k^2 = \omega^2\mu_0\epsilon - i\omega\mu_0\sigma \tag{3.1}$$

so that previous analysis would be applicable except that the propagation constant for a Debye medium is completely determined by expressing the permittivity as

$$\epsilon = \epsilon_0\left[\left(\frac{(\epsilon^* - \epsilon_\infty)}{1 + (\omega\tau)^2}\right) + \epsilon_\infty\right] \tag{3.2}$$

where ϵ^* and ϵ_∞ are constants and τ represents a characteristic relaxation time, and where, using these constants, the human tissue propagation constant (3.1) is defined in terms of (3.1) and the Debye medium conductivity σ which is defined by the relation,

$$\sigma = \left(\frac{(\epsilon^* - \epsilon_\infty)}{1 + (\omega\tau)^2}\right)\omega^2\tau\epsilon_0 \tag{3.3}$$

The only difference here is that the branch cuts used to represent the propagation constant on a Riemann surface for a pure Debye medium material are all along the imaginary axis.

4 Surface Integral Equation Methods

In this section we shall show how in the case where the irradiated structure consists of homogneous regions which are delimited by diffeomorphisms of the interior of a sphere or a torus in three dimensional space (in the body of revolution case) to represent the solution of the scattering problem as the solution of two combined field integral equations with integral operators formed from from the Green's functions defined on opposite sides of the separating surfaces. The surface integral equation methods reduce the computational complexity in the sense that they require discretization electric and magnetic fields defined on a surface rather than on a region of three dimensional space. In a general nonrotationally symmetric setting the development which follows is valid for regions which are the interior of diffeomorphisms of N handled spheres.

4.1 Combined Field Integral Equations

Consider a set Ω in $\mathbb{R}^3$ with boundary surface $\partial\Omega$ on which are induced electric and magnetic surface currents $\vec{J}_j$ and $\vec{M}_j$. If we have a simple $N+1$ region problem, where we have N inside and a region outside all N bounded homogenous aerosol particles corresponds to the region index j being equal to 1 and the region inside corresponds to j values ranging from 2 to $N+1$, then if the propagation constant k_j in region j is defined also by a function k_j, naturally defined on a Riemann surface as the square root of,

$$k_j^2 = \omega^2\mu\epsilon - i\omega\mu\sigma \tag{4.1}$$

For a Debye medium (Daniel, [11]) the branch cuts are along the imaginary ω axis. For a Lorentz medium particle (Brillouin, [4], [29]) the branch cuts are in the upper half of the complex ω plane parallel to the real axis. where μ, ϵ, and σ are functions of frequency that assure causality and that the radiation does not travel faster than the speed of light in vacuum. There are two Helmholtz equations, one for the interior of the particle and the other for the exterior, defined by

$$(\Delta + k_j^2)G_r = 4\pi\delta \tag{4.2}$$

where G_j is the temparate, rotationally invariant, fundamental solution ([16]) of the Helmholtz operator. We let

$$J_1 = J = -J_2 \tag{4.3}$$

and

$$M_1 = M = -M_2 \tag{4.4}$$

where we assume that the surface $S_{(1,2)}$ separates region 1 and region 2. We generalize equations (4.3) and (4.4) inductively by saying that for any surface $S_{(j,\tilde{j})}$ separating region j from region $\tilde{j}$ where

$$j < \tilde{j} \tag{4.5}$$

we have

$$J_j = J = -J_{\tilde{j}} \tag{4.6}$$

and

$$M_j = M = -M_{\tilde{j}} \tag{4.7}$$

We define

$$\mathcal{I} = \left\{(j,\tilde{j}) : S_{(j,\tilde{j})} \text{ is a separating surface}\right\} \tag{4.8}$$

where j is less than $\tilde{j}$. We get a single coupled, combined field integral equation which describes the interaction of radiation with the conglomerate aerosol particle or cluster given by

$$\begin{aligned}\vec{n}\times\vec{E}^{inc} = \vec{n}\times\sum_{(j,\tilde{j})\in\mathcal{I}}\Bigg\{&\left(\frac{i\omega}{4\pi}\right)\int_{S_{(j,\tilde{j})}}\int \vec{J}(\bar{r})\left(\mu_j\cdot G_j(r,\bar{r}) + \mu_{\tilde{j}}\cdot G_{\tilde{j}}(r,\bar{r})\right)da(\bar{r})\\ &+\frac{i}{4\pi\omega}grad\left\{\int_{S_{(j,\tilde{j})}}\int(div_s\cdot\vec{J})\left[\frac{G_j(r,\bar{r})}{\epsilon_j} + \frac{G_{\tilde{j}}(r,\bar{r})}{\epsilon_{\tilde{j}}}\right]da(\bar{r})\right\} +\\ &\left(\frac{1}{4\pi}\right)curl\left(\int_{S_{(j,\tilde{j})}}\int\vec{M}(\bar{r})\cdot\left(G_j(r,\bar{r}) + G_{\tilde{j}}(r,\bar{r})\right)da(\bar{r})\right)\Bigg\}\end{aligned} \tag{4.9}$$

In addition to equation (4.9) we need equation involving the magnetic vector H^{inc} of the stimulating electromagnetic field which is given by

$$\begin{aligned}\vec{n}\times\vec{H}^{inc} = \vec{n}\times\sum_{(j,\tilde{j})\in\mathcal{I}}\Bigg\{&\left(\frac{i\omega}{4\pi}\right)\int_{S_{(j,\tilde{j})}}\int \vec{M}(\bar{r})\left(\epsilon_1\cdot G_j(r,\bar{r}) + \epsilon_2\cdot G_{\tilde{j}}(r,\bar{r})\right)da(\bar{r})\\ &+\left(\frac{i}{4\pi\omega}\right)grad\left\{\int_{S_{(j,\tilde{j})}}\int(div_s\cdot\vec{M})\left[\frac{G_j(r,\bar{r})}{\mu_j} + \frac{G_{\tilde{j}}(r,\bar{r})}{\mu_{\tilde{j}}}\right]da(\bar{r})\right\} +\\ &\frac{1}{4\pi}curl\left(\int_{S_{(j,\tilde{j})}}\int\vec{J}(\bar{r})\left(G_j(r,\bar{r}) + G_{\tilde{j}}(r,\bar{r})\right)da(\bar{r})\right)\Bigg\}\end{aligned} \tag{4.10}$$

Once the coupled combined field system (4.9) and (4.10) is solved for $\vec{J}$ and $\vec{M}$, the surface electric and magnetic currents respectively and we define the surface electric charge density by ([13], p 7)

$$\rho^e(\bar{r}) = \frac{i}{\omega}\left[div_s\cdot\vec{J}(\bar{r})\right] \tag{4.11}$$

and the surface magnetic charge density

$$\rho^m(\bar{r}) = \frac{i}{\omega}\left[div_s \cdot \vec{M}(\bar{r})\right] \tag{4.12}$$

where div_s is the surface divergence. Now for each region index j we define

$$\mathcal{J}(j) = \left\{\tilde{j} : (j, \tilde{j}) \in \mathcal{I}\right\} \tag{4.13}$$

where $\mathcal{I}$ is the set of all indices of separating surfaces defined by (4.8). We now need to be able to express the electric and magnetic fields inside and outside the scattering body. We first define the vector potentials $\vec{A}_j$ and $\vec{F}_j$ by the rules, ([13] [21])

$$\vec{A}_j = \sum_{\tilde{j}\in\mathcal{J}(j)} \left[\frac{\mu_j}{4\pi}\int_{S_{(j,\tilde{j})}}\int \vec{J}_j(\bar{r}) \cdot G_j(r,\bar{r})da(\bar{r})\right] \tag{4.14}$$

$$\vec{F}_j = \sum_{\tilde{j}\in\mathcal{J}(j)} \left[\left(\frac{\epsilon_j}{4\pi}\right)\int_{S_{(j,\tilde{j})}}\int \vec{M}_j(\bar{r}) \cdot G_j(r,\bar{r})da(\bar{r})\right] \tag{4.15}$$

The scalar potentials are defined in terms of the electric charge density (4.11) and magnetic charge density (4.12) by the rules,

$$\Phi_j(\bar{r}) = \sum_{\tilde{j}\in\mathcal{J}(j)} \left[\left(\frac{1}{4\pi\epsilon_j}\right)\int_{S_{(j,\tilde{j})}}\int \rho_j^e(\bar{r})G_j(r,\bar{r})da(\bar{r})\right] \tag{4.16}$$

and

$$\Psi_j(\bar{r}) = \sum_{\tilde{j}\in\mathcal{J}(j)} \left[\left(\frac{1}{4\pi\mu_j}\right)\int_{S_{(j,\tilde{j})}}\int \rho_j^m(\bar{r})G_j(r,\bar{r})da(\bar{r})\right] \tag{4.17}$$

We now can define the electric and magnetic vectors inside the region j in terms of these potentials (4.14), (4.15), (4.16), and (4.17) by the rules,

$$\vec{E}_j = -i\omega\vec{A}_j(r) - grad(\Phi_j(r) + \frac{1}{\epsilon_j}curl(\vec{F}_j)(r) \tag{4.18}$$

and

$$\vec{H}_j = -i\omega\vec{F}_j(r) - grad(\Psi_j(r) + \frac{1}{\mu_j}curl(\vec{A}_j)(r) \tag{4.19}$$

Similar equations apply outside the body, by there the fields represented are the differences $\vec{E}_1^s$ and $\vec{H}_1^s$ between the total electric and magnetic vectors and the electric vector $\vec{E}^{inc}$ and the magnetic vector $\vec{H}^{inc}$ of the incoming wave that is providing the stimulation. Thus ([13]) we see that outside the body,

$$\vec{E}_1^s = -i\omega\vec{A}_1(r) - grad(\Phi_1(r) + \frac{1}{\epsilon_1}curl(\vec{F}_1)(r) \tag{4.20}$$

and

$$\vec{H}_1^s = -i\omega\vec{F}_1(r) - grad(\Psi_1(r) + \frac{1}{\mu_1}curl(\vec{A}_1)(r) \tag{4.21}$$

These equations generalize the formulation of Glisson ([13]) to a three dimensional structure whose regions of homogeneity are diffeomorphisms of the interior of the sphere or a torus in R^3. If the scattering structure is not a body of revolution, then the region may be a diffeomorph of an N handled sphere.

5 Zeros of Functions of a Complex Variable

Important design problems can be solved with good algorithms for finding zeros of entire or meromorphic functions of a complex variable. One of the most important problems attached to Riemann's name was the Riemann hypothesis. In this section we discuss some novel homotopy methods ([6]) for finding zeros of analytic functions. The problem of finding modes of propagation in an anisotropic, magnetically lossy coating on a perfect conductor ([10]) is related to the problem of finding complex numbers z such that

$$cosh(\sqrt{z}) - \frac{A \cdot z + B}{C \cdot z + B} = 0 \tag{5.1}$$

by moving this problem up to a higher algebra where the solution becomes transparent and then following a homotopy path down to the solutions in the space of interest; this permitted the authors to track propagation constants as magnetic properties went through regions of anomalous dispersion and the material thickness changed.

We consider here the problem of finding complex numbers z such that $sin(z)$ is equal to z. Since there are no polynomial functions $P(z)$ and entire functions $h(z)$ such that

$$-\frac{i}{2}exp(iz) + \frac{i}{2}exp(-iz) - z = P(z)exp(h(z)) \tag{5.2}$$

it is clear from the theorem of Picard that there are an infinite number of solutions of the equation

$$sin(z) - z = 0 \tag{5.3}$$

We transform the equation,

$$sin(z) = z \tag{5.4}$$

to an equation in another space by using auxiliary functions so that the transformed equation has the form,

$$sin(A(s)z(s)) = (z(s) + B(s)) \tag{5.5}$$

where

$$B(s) = (n \cdot \pi i)(1 - s) \tag{5.6}$$

and

$$z(0) = -n\pi i \tag{5.7}$$

and

$$A(s) = i(1 - s) + s \tag{5.8}$$

so that when s is equal to zero, equation (5.5) has the form,

$$sin(i(-n\pi i)) = sin(n\pi) = (-n\pi i + n\pi i) \tag{5.9}$$

which is true, and when s is equal to 1, then as the trivial equation (5.9) holds at one end of the homotopy path and if equation (5.5) is preserved all the way along the path, and as this equation has the form

$$sin(A(1) \cdot z) = z + B(1) \tag{5.10}$$

at s equal to one, since

$$A(1) = 1 \tag{5.11}$$

and

$$B(1) = 0 \tag{5.12}$$

we see that we obtain a solution of equation (5.4) at the other end of the path.

Thus, the problem is finding a scheme for assuring that the equation (5.5) is preserved all the way along the path. Differentiating both sides of equation (5.5) we see that

$$\begin{aligned} z'(s) + B'(s) &= \\ cos(A(s)z(s))\,\{A'(s)\cdot z(s) + A(s)\cdot z'(s)\} & \end{aligned} \tag{5.13}$$

Collecting terms involving $z'(s)$ we find that

$$\begin{aligned} \{A(s)cos(A(s)z(s)) - 1\}\, z'(s) &= \\ B'(s) - z(s)A'(s)cos(A(s)z(s)) & \end{aligned} \tag{5.14}$$

which leads, after solving equation (5.14), to a coupled system of differential equations in $x(s)$ and $y(s)$ with known values at $s = 0$. Thus,

$$x'(s) = Real\left\{\frac{B'(s) - z(s)A'(s)cos(A(s)z(s))}{A(s)cos(A(s)z(s)) - 1}\right\} \tag{5.15}$$

and

$$y'(s) = Imag\left\{\frac{B'(s) - z(s)A'(s)cos(A(s)z(s))}{A(s)cos(A(s)z(s)) - 1}\right\} \tag{5.16}$$

where

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -n\pi \end{pmatrix} \tag{5.17}$$

These equations have been computer tested and orbits starting at $x(0) + iy(0)$ equal to

$$x(0) + iy(0) = 0 + 2\pi i, \tag{5.18}$$

for example, end up at

$$z = 7.4976\cdots - i2.7636\cdots \tag{5.19}$$

6 Applications

The homotopy method described in the last section provides us with a powerful design tool. By designing parameters in an irradiated toroidal plasma that will increase its efficiency of energy absorption by a factor of 1,000,000 one could in the light of the already successful Fusion reaction in England design practical, commercial fusion reactors which would replace all other means of generating power, abate the global warming, and give us a means of having safe drinking water 10 years from now.

With the careful design of irradiating microwave systems, one could focus microwave energy in the interior of the body on a cancer tumor; if one could highly locally raise the temperature of the cancer tumor by just 4° C one would kill the tumor without harming nearby normal tissue. With the surface integral equation approach described in the previous section, one could make detailed models of the lungs, spleen, liver, kidneys et cetera and have the first realistic model which predicts the internal detail of the response of a man to a complex electromagnetic field. The only exact solutions that have previously been used to attempt to validate electromagnetic interaction codes have been those for spherical scatterers. Since half of the effort in solving integral equations of electromagnetic scattering involves finding entries in a matrix operator representing the discretization of the integral equations (4.9) and (4.10), the exact formulas (2.13) and (2.18) for integrals of functions defined on Riemann surfaces have made it easier to accurately determine the interaction of electromagnetic radiation with penetrable bodies having rotational symmetry; this will permit a more rigorous validation of the computer code predicting the detailed internal response of the human body to a complex radiation field.

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A THEORY OF HEATING OF VOIGT SOLIDS AND FLUIDS BY EXTERNAL ENERGY SOURCES

D. K. Cohoon
43 Skyline
Glen Mills, PA 19342

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The purpose of this paper is to develop both (i) a theory of laser stimulated vaporization of droplets and (ii) a theory of internal heating resulting from vibration waves in linearly responding elastic material. There are applications to sending information through clouds on laser beams and to the control of temperature in ultrasonic welding.

We develop a theory of thermal excursions resulting from ultrasonic welding, and interpret it as an elastic interaction with damping in a Voigt solid. It is hypothesized that with good control of temperature, one could achieve strong and uniform welds by this process and greatly reduce the cost of manufacture of aircraft, and other aluminum structures. We consider equations describing the conservation of mass, momentum, and energy coupled by an equation of state, and consider general mass, momentum, and energy transfer relationships in a compressible body subjected to external stimuli. For the Voigt solid theory, a linear elastic theory with damping forces, we show how some simple local time averaging gives us a dovetailed system consisting of the elastic wave equations whose solution provides the source term for a heat equation. For the more general theory of droplet vaporization we illustrate a general nonlinear energy equation which includes a radiation energy conductivity term.

Contents

1 INTRODUCTION

We use this concept of a material derivative and fluxes of mass, momentum, kinetic energy, internal energy, temperature, and radiation to express the conservation of mass, momentum, and energy in a Voigt solid that is stimulated by an elastic wave energy source. Initially there are more dependent variables than there are equations. However, these equations are coupled by an equation of state which enables one to develop a semigroup formulation which will predict pressure, density, velocity, and temperature distributions in the interior of the stimulated solid. Local time averaging gives us a heat equation with an elastic energy source term.

1.1 Vector Analysis

The material derivative of a function f is defined by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \tag{1.1.1}$$

Thus, the material derivative is, if we define,

$$\vec{v} = \frac{\partial x}{\partial t}\vec{e}_x + \frac{\partial y}{\partial t}\vec{e}_y + \frac{\partial z}{\partial t}\vec{e}_z \tag{1.1.2}$$

given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot grad) \tag{1.1.3}$$

where $\vec{v}$ is the velocity of a point in the fluid.

1.2 External Energy Sources

Thermal energy is transferred by conductivity and internal radiation as well as radiation from the surface. We assume that the elastic material is electromagnetically polyanisotropic.

a material more general than a bianisotropic material. The nonlinear Faraday Maxwell equation is given by

$$curl(\vec{E}) = \mathcal{F}\left(\vec{E}, \vec{H}, \cdots, \left(\frac{\partial}{\partial t}\right)^k \vec{E}, \left(\frac{\partial}{\partial t}\right)^k \vec{H}, \cdots\right) \tag{1.2.1}$$

while the nonlinear Ampere Maxwell equation has the form

$$curl(\vec{H}) = \mathcal{G}\left(\vec{E}, \vec{H}, \cdots, \left(\frac{\partial}{\partial t}\right)^k \vec{E}, \left(\frac{\partial}{\partial t}\right)^k \vec{H}, \cdots\right) \tag{1.2.2}$$

For a general material where we have continuity of tangential components of $\vec{E}$ and $\vec{H}$ across the boundary separating regions of continuity of electromagnetic properties the radiation source term is

$$\left(\frac{\partial}{\partial t}\right) Q_{in} = (1/2)Re(div(\vec{E} \times \vec{H}^*) \tag{1.2.3}$$

The radiation source term which provides a thermal energy is for a linearly responding material given by

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right) Q_{in} = \\ (1/2)Re\left\{\vec{E}^* \cdot (i\omega\overline{\overline{\epsilon}} + \overline{\overline{\sigma}})\vec{E} + \vec{E} \cdot \overline{\overline{\alpha}}^* \vec{H}^* - \right. \\ \vec{H} \cdot (i\omega\overline{\overline{\mu}}^* \vec{H}^*) + \vec{H} \cdot (\overline{\overline{\beta}}^* \vec{E}^*) + \\ \left. \chi_{\partial\Omega}(r)\sigma_s \mid \vec{E}_{tangential} \mid^2 \right\}\end{aligned} \tag{1.2.4}$$

where if $\partial\Omega$ is the surface containing the impedance sheet, then

$$\int_\Omega \chi_{\partial\Omega}\sigma_s \mid \vec{E}_{tangential} \mid^2 dv = \int_{\partial\Omega} \sigma_s \mid \vec{E}_{tangential} \mid^2 dA \tag{1.2.5}$$

defines the characteristic function $\chi_{\partial\Omega}$ of the surface supporting the impedance sheet, ϵ is the permittivity, μ is the permeability, σ denotes conductivity, and $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ are coupling tensors which appear in the linear Faraday and Ampere Maxwell equations, respectively.

Another internal source term is the Voigt solid damping term contribution which will be derived in the sections which follow. Another source of heat is the friction of a mechanical vibrator on the surface of the aluminum.

2 Mass Transfer

We consider that through melting or movement of a fluid that matter can flow across the boundary of a surface or that, in the case of an elastic medium, that it cannot, and examine the consequences and mathematical representation of of these assumptions.

2.1 Continuity Equation

Assuming that in the Voigt solid or liquid interior that the rate at which mass is created or destroyed is given by Q_M and that the flux of mass across a surface is given by $\rho\vec{v}$ we see that

$$\frac{\partial\rho}{\partial t} + div(\rho\vec{v}) = Q_M \tag{2.1.1}$$

or if $Q_M = 0$ that

$$div(\rho\vec{v}) = -\frac{\partial\rho}{\partial t} \tag{2.1.2}$$

3 Momentum Equations

The conservation of momentum is our most fundamental law of physics. We examine the consequences of this for the Voigt solid and for liquids.

3.1 Voigt Solid Momentum Conservation

In this section we derive the conservation of momentum by equating the rate of change of momentum to the work done by the fluid pressure and the viscous forces and the body forces and the flux of momentum across the boundaries of test volumes. We define the velocity as

$$\vec{v} = u\vec{e}_x + v\vec{e}_y + w\vec{e}_z \tag{3.1.1}$$

An important identity involving the dyadic product of two vectors $\vec{A}$ and $\vec{B}$ is

$$div(\vec{A}\vec{B}) = div(\vec{A})\vec{B} + (\vec{A}\cdot grad)\vec{B} \tag{3.1.2}$$

Another important quantity is the tensor or dyadic quantity quantity obtained by taking the gradient of a vector field given by

$$grad(\vec{A}) = \sum_{i=1}^{n}\left(\frac{\partial A_j}{\partial x_i}\right)\vec{e}_i\vec{e}_j \tag{3.1.3}$$

Using equation (3.1.3) we define the symmetric strain tensor in terms of the displacement $\vec{U}$ of a point of a solid from its equilibrium position as

$$\begin{aligned}\bar{\bar{e}} &= \frac{grad(\vec{U}) + grad(\vec{U})^t}{2}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{2}\right)\left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j}\right)\vec{e}_i\vec{e}_j\end{aligned} \tag{3.1.4}$$

and cubical dilatation θ is given by

$$\theta = div(\vec{U}) = \sum_{i=1}^{n}\left(\frac{\partial U_j}{\partial x_i}\right) \tag{3.1.5}$$

The Voigt solid elastic stress tensor is defined by

$$\overline{\overline{S}} =$$

$$2 \cdot \mu \overline{\overline{e}} \; + \; \lambda \theta \overline{\overline{I}} \; + \; 2 \cdot \tilde{\mu} \frac{\partial \overline{\overline{e}}}{\partial t} \; + \; \tilde{\lambda} \frac{\partial \theta}{\partial t} \overline{\overline{I}} \tag{3.1.6}$$

where

$$\overline{\overline{I}} \; = \; \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\delta_{(i,j)} \vec{e}_i \vec{e}_j \right) \tag{3.1.7}$$

If $\vec{F}$ is the force per unit mass, and ρ is the mass per unit volume, then the generalized equations of elasticity for a stress tensor $\overline{\overline{S}}$ by Newton's force is equal to mass times acceleration law, or

$$\rho \frac{\partial^2 \vec{U}}{\partial t^2} \; = \; \rho \vec{F} \; + \; div(\overline{\overline{S}}) \tag{3.1.8}$$

When the stress tensor $\overline{\overline{S}}$ is given by the Voigt solid relationship (3.1.6)

$$\rho \frac{\partial^2 \vec{U}}{\partial t^2} \; = \; \rho \vec{F} \; +$$

$$(\lambda + \mu) grad(\theta) \; + \; \mu \Delta \vec{U} \; + \; (\tilde{\lambda} \; + \; \tilde{\mu}) \frac{\partial}{\partial t} grad(\theta) \; + \; \tilde{\mu} \Delta \frac{\partial \vec{U}}{\partial t} \tag{3.1.9}$$

where in Cartesian coordinates the Laplacian Δ is defined by

$$\Delta \vec{U} \; = \; \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) \vec{U} \tag{3.1.10}$$

When the material through which the elastic wave is propagating is three or seven dimensional, the displacement vector $\vec{U}$ is necessarily a curl plus a gradient given by

$$\vec{U} \; = \; grad(\phi) \; + \; curl(\vec{\psi}) \tag{3.1.11}$$

This is true for any C^∞ function defined on an open set in $\mathbf{R}^n$ with values in $\mathbf{C}^n$ for n equal to three or seven, and can be seen from the following lemma ([19]).

Lemma 3.1 *If n is three or seven, then for every open set Ω in $\mathbf{R}^n$ and for every vector field $\vec{F}$ in $C^\infty(\Omega, \mathbf{C}^n)$ there is a vector field $\vec{G}$ in the same space such that*

$$\vec{F} = grad(div(\vec{G})) + curl(curl(-\vec{G})) \tag{3.1.12}$$

where if n is equal to seven the curl is defined by the rule,

$$curl(\vec{E}) =$$

$$\sum_{i=1}^{7} \left[\left(\frac{\partial E_{i+3}}{\partial x_{i+1}} - \frac{\partial E_{i+1}}{\partial x_{i+3}} \right) + \right.$$

$$\left. \left(\frac{\partial E_{i+6}}{\partial x_{i+2}} - \frac{\partial E_{i+2}}{\partial x_{i+6}} \right) + \left(\frac{\partial E_{i+5}}{\partial x_{i+4}} - \frac{\partial E_{i+4}}{\partial x_{i+5}} \right) \right] \vec{e}_i \tag{3.1.13}$$

where $\vec{e}_i$ is the unit vector in the direction of the ith coordinate axis in 7 dimensional space and

$$E_{i+7} = E_i \tag{3.1.14}$$

If we then substitute equation (3.1.11) into (3.1.11) we deduce that

$$grad\left(\rho\frac{\partial^2\phi}{\partial t^2} - (\lambda+2\mu)\Delta\phi - (\tilde{\lambda}+2\tilde{\mu})\Delta\frac{\partial\phi}{\partial t}\right)$$

$$= curl\left(\mu\Delta\vec{\psi} + \tilde{\mu}\Delta\frac{\partial\vec{\psi}}{\partial t} - \rho\frac{\partial^2\psi}{\partial t^2}\right) \tag{3.1.15}$$

where Δ is defined by (3.1.10). If we take the dot product of both sides of equation (3.1.15) with the gradient of any test function P with compact support and integrate over an open set containing the support of this test function, then the curl term disappears, since the curl of a gradient is the zero vector. We get two wave equations with damping terms and different wave speeds satisfied by ϕ and $\vec{\psi}$. The ϕ wave equation is

$$\rho\frac{\partial^2\phi}{\partial t^2} = (\lambda+2\nu)\Delta\phi - (\tilde{\lambda}+2\cdot\tilde{\mu})\Delta\frac{\partial\phi}{\partial t} \tag{3.1.16}$$

and

$$\rho\frac{\partial^2\vec{\psi}}{\partial t^2} = \mu\Delta\vec{\psi} + \tilde{\mu}\Delta\frac{\partial\vec{\psi}}{\partial t} \tag{3.1.17}$$

with Δ being defined by (3.1.10). Note that if we set $\tilde{\mu}$ and $\tilde{\lambda}$ equal to zero, then we get exactly the wave equations for the two types of observed Earthquake waves. If we Fourier transform all terms of equations (3.1.16) and (3.1.17) with respect to time we see that the Fourier transforms of both $\vec{\psi}$ and ϕ with respect to time satisfy a Helmholtz equation of the form,

$$\Delta V + k^2 V = 0 \tag{3.1.18}$$

where Δ is the Laplacian defined by (3.1.10) and k is a complex constant. Thus, except for the rather complex boundary conditions these equations might be solved by standard theories. The boundary conditions are highly mixed and require us to consider

- a region of welded contact between the plates where both the displacement and the stress tensor are continuous,
- a free surface where all the entries of the stress tensor are zero,
- a region of contact of the vibrator and the surface of the material being welded where the stress tensor is specified,
- the nonwelded contact region where the normal components of the stress and displacement are continuous, and
- the region of contact of the workpiece and the clamp, where the normal components of the stress are specified and the normal component of the displacement is fixed at zero.

3.2 Generalized Navier Stokes Equations

For compressible materials, the momentum conservation equations are nonlinear. The momentum flux is the dyad $\rho\vec{v}\vec{v}$ and using the concept of conservation of mass or equation (2.1.2) and equation (3.1.2) we see that

$$div(\rho\vec{v}\vec{v}) = div(\rho\vec{v}) + \rho(\vec{v}\cdot grad)\vec{v}$$

$$= -\frac{\partial\rho}{\partial t}\vec{v} + \rho(\vec{v}\cdot grad)\vec{v} \tag{3.2.1}$$

If p is the pressure, then the total stress tensor Π is given by

$$\overline{\overline{\Pi}} = -p(\vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z) + \overline{\overline{\tau}} \tag{3.2.2}$$

The viscous stress tensor is given, using equation (3.1.1) for velocity, by the rule,

$$\overline{\overline{\tau}} =$$

$$\mu\left(2\frac{\partial u}{\partial x}\right)\vec{e}_x\vec{e}_x + \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\vec{e}_y\vec{e}_x +$$

$$\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\vec{e}_z\vec{e}_x + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\vec{e}_x\vec{e}_y$$

$$+\mu\left(2\frac{\partial v}{\partial y}\right)\vec{e}_y\vec{e}_y + \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\vec{e}_z\vec{e}_y +$$

$$\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\vec{e}_x\vec{e}_z + \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\vec{e}_y\vec{e}_z$$

$$+\mu\left(2\frac{\partial w}{\partial z}\right)\vec{e}_z\vec{e}_z - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\vec{e}_x\vec{e}_x$$

$$-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\vec{e}_y\vec{e}_y$$

$$-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\vec{e}_z\vec{e}_z \tag{3.2.3}$$

We have seen that the total stress tensor, equation (3.2.2) is given in terms of the pressure p and the viscous stress (3.2.3). The momentum equation is given by

$$\frac{\partial}{\partial t}(\rho\vec{v}) = -div(\rho\vec{v}\vec{v})$$

$$\rho\vec{f} + div(\overline{\overline{\Pi}}) \tag{3.2.4}$$

Using equation (3.2.1) we see that equation (3.2.4) and equations (3.2.2) and (3.2.3) we see that

$$\rho\frac{\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot grad(\vec{v})) =$$

$$\rho \vec{f} - grad(p) + div(\overline{\overline{\tau}}) \tag{3.2.5}$$

Using the concept of material derivative, equation (1.1.1) and assuming that $\vec{f}$ is the zero vector, equation (3.2.5) reduces to

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}grad(p) + \frac{1}{\rho}div(\overline{\overline{\tau}}) \tag{3.2.6}$$

4 Energy Conservation

There is internal energy, kinetic energy, work done by the viscous forces (equation 3.2.3), pressure, and work done by the external body forces. The energy is transferred from one region of the heated Voigt solid to another by thermal conduction, kinetic energy flux, and radiation conduction processes, and by the external elastic and thermal energy source. For boiling liquids we consider viscous dissipation functions and a radiation conductivity term.

4.1 A Heat Equation for Voigt Solids

We begin by considering the Voigt solid stress tensor and then go on to analysis of energy transfer where viscous dissipations functions are responsible for energy transfer.

We now consider specific energy per unit mass e within a stimulated Voigt solid, and we let the velocity $\vec{V}$ of a point be defined by

$$\vec{V} = \frac{\partial \vec{U}}{\partial t} \tag{4.1.1}$$

where $\vec{U}$ is the displacement from equilibrium. Then the total energy within a volume Ω is given by

$$\mathcal{E}_\Omega(t) = \int_\Omega \left\{\rho(e + \vec{V}\cdot\vec{V}/2)\right\} dv \tag{4.1.2}$$

The time derivative of $\mathcal{E}_\Omega(t)$ is the rate of energy input into Ω by

- body forces,
- the stress system,
- the flux of kinetic energy across the boundaries,
- thermal heat conduction,
- internal heat generation,
- radiative transport, and
- internal energy flux.

The above means of energy transport are all important in fluid flow, but in elastic media, many of the terms may be ignored because there is no gross motion of material across boundaries. With our periodicity assumption, many of the terms which are conceptually small will be shown to vanish exactly when they are smoothed by using local time averages. This local smoothing may be thought of as a transistion from a temporally microscopic to a temporally macroscopic theory.

To get to the final form of the equation that we consider we shall assume that the integral of the product of a slowly varying function and a highly oscillatory function is nearly zero. The rigorous energy equation may be expressed in the form,

$$\frac{d}{dt}\left(\int_\Omega(\rho(e+\vec{V}\cdot\vec{V}/2))dv\right) =$$

$$\int_\Omega(\vec{F}\cdot\vec{V} + div(\overline{\overline{S}}\cdot\vec{V}) - (div(\rho(\vec{V}\cdot\vec{V}/2)\vec{V}))$$

$$+\ div(\overline{\overline{K}}\cdot grad(T)) + \frac{\partial Q}{\partial t} + div(q_r - e\rho\vec{V}))dv \tag{4.1.3}$$

where the terms on the right side of (4.1.3) are respectively

- power transfer by body forces
- rate of kinetic energy transfer across the boundary
- the rate of energy transfer by thermal conduction
- the rate at which energy is created internally
- the rate at which energy is transferred into the body by radiation,
- the rate at which internal energy is transferred across the boundary by material motion.

From equation (4.1.3) we deduce an energy transfer equation,

$$\left(\frac{\partial\rho}{\partial t}\right)e + \rho\left(\frac{\partial e}{\partial t}\right) + \left(\frac{\partial\rho}{\partial t}\right)\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) + \rho\vec{V}\cdot\left(\frac{\partial\vec{V}}{\partial t}\right) =$$

$$\vec{F}\cdot\vec{V} + div(\overline{\overline{S}}\cdot\vec{V}) - div(\rho\left(\frac{\vec{V}\cdot\vec{V}}{2}\right)\vec{V}) + div(\overline{\overline{K}}\cdot grad(T))$$

$$\frac{\partial Q}{\partial t} + div(q_r) - ediv(\rho\vec{V}) - \rho\vec{V}\cdot grad(e) \tag{4.1.4}$$

If a slowly varying time envelope is riding on a rapidly varying oscillation, (e.g. very rapid vibrations and a periodic movement of the source of those vibrations or a steady increase in temperature resulting from those vibrations) then we can use the local time averaging operator

$$P_{T_p}(f)(t) = \overline{f}(t) = \left(\frac{1}{T_p}\right)\int_{t-T_p}^{t} f(\tau)d\tau \tag{4.1.5}$$

then it is clear that

Lemma 4.1 *If f is periodic with period T_p and if P_{T_p} is defined by (4.1.5) then for all real t*

$$P_{T_p}(f \cdot f') = 0 \tag{4.1.6}$$

This follows from the fact that

$$(f \cdot f')(t) = \frac{d}{dt}\left(\frac{f^2}{2}\right)(t) \tag{4.1.7}$$

and the fact that if f is periodic with period T_p that then

$$f^2(t + T_p) - f^2(t) = 0 \tag{4.1.8}$$

We shall use elementary vector analysis to reduce the energy equation (4.1.4) to a place where we can use the Lemma and the local time average operation P_{T_p} defined by equation (4.1.5) to get a simplified heat equation.

We shall use the identity,

$$\begin{aligned} div((\vec{A}\cdot\vec{B})\vec{C}) &= (\vec{A}\cdot\vec{B})div(\vec{C}) + \vec{C}\cdot grad(\vec{A}\cdot\vec{B}) = \\ &(\vec{A}\cdot\vec{B})div(\vec{C}) + \vec{C}\cdot\left\{A\times curl(\vec{B}) + \vec{B}\times curl(\vec{A})\right. \\ &\left. + (\vec{A}\cdot grad)\vec{B} + (\vec{B}\cdot grad)\vec{A}\right\} \end{aligned} \tag{4.1.9}$$

which means that if we let $\vec{A}$ be equal to $\vec{B}$ be equal to $\vec{C}$ be equal to $\rho\vec{V}$ to deduce a simplification of the divergence of ρ times half of the dot product of $\vec{V}$ with itself times $\vec{V}$. We see that

$$\begin{aligned} div(\left(\rho\frac{\vec{V}\cdot\vec{V}}{2}\right)\vec{V}) &= \left(\frac{\vec{V}\cdot\vec{V}}{2}\right)div(\rho\vec{V}) + \vec{V}\cdot grad(\vec{V}\cdot\vec{V}) = \\ &\left(\frac{\vec{V}\cdot\vec{V}}{2}\right)div(\rho\vec{V}) + \rho\frac{\vec{V}}{2}\cdot\left\{\vec{V}\times curl(\vec{V}) + \right. \\ &\left.\vec{V}\times curl(\vec{V}) + (\vec{V}\cdot grad)\vec{V} + (\vec{V}\cdot grad)\vec{V}\right\} \end{aligned} \tag{4.1.10}$$

But since

$$\vec{V}\cdot(\vec{V}\times curl(\vec{V})) = 0, \tag{4.1.11}$$

we see that equation (4.1.10) reduces to

$$- div(\left(\rho\frac{\vec{V}\cdot\vec{V}}{2}\right)\vec{V}) = \left(\frac{\vec{V}\cdot\vec{V}}{2}\right)\frac{\partial\rho}{\partial t} - \rho\vec{V}\cdot((\vec{V}\cdot grad)\vec{V}) \tag{4.1.12}$$

The generalized momentum conservation equation,

$$\rho\frac{\partial\vec{V}}{\partial t} = -\rho(\vec{V}\cdot grad)\vec{V} + \vec{F} + div(\overline{\overline{S}}) \tag{4.1.13}$$

Using the equation (4.1.12), which simplifies the divergence of the kinetic energy flux, and the result

$$\rho\vec{V}\cdot\frac{\partial\vec{V}}{\partial t} = -\rho\vec{V}\cdot(\vec{V}\cdot grad)\vec{V} + \vec{F}\cdot\vec{V} + div(\overline{\overline{S}})\cdot\vec{V} \qquad (4.1.14)$$

of dotting all terms generalized momentum equation, (4.1.13), we deduce from (4.1.3) that

$$\frac{\partial\rho}{\partial t}e + \rho\frac{\partial e}{\partial t} + \frac{\partial\rho}{\partial t}\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) +$$

$$\left\{-\rho\vec{V}\cdot((\vec{V}\cdot grad)\vec{V} + \vec{F}\cdot\vec{V} + div(\overline{\overline{S}})\cdot\vec{V}\right\}$$

$$= \vec{F}\cdot\vec{V} + div(\overline{\overline{S}}\cdot\vec{V}) +$$

$$\left\{\left(\frac{\vec{V}\cdot\vec{V}}{2}\right)\frac{\partial\rho}{\partial t} - \rho\vec{V}\cdot((\vec{V}\cdot grad)\vec{V}\right\} +$$

$$div(\overline{\overline{K}}\cdot grad(T)) + \frac{\partial Q}{\partial t} + div(q_r)$$

$$- e(div(\rho\vec{V})) + \rho\vec{V}\cdot grad(e) \qquad (4.1.15)$$

where the terms in energy equation (4.1.15) that differ from the original energy equation (4.1.3) are enclosed in curly brackets. We further simplify equation (4.1.15) by using the mass conservation equation (2.1.2) and cancelling out terms that appear on both sides of the equal sign of equation (4.1.15) to obtain,

$$\rho\left(\frac{\partial\rho}{\partial t}\right) + div(\overline{\overline{S}})\cdot\vec{V} = div(\overline{\overline{S}}\cdot\vec{V}) +$$

$$div(\overline{\overline{K}}\cdot grad(T)) + \frac{\partial Q}{\partial t} + div(q_r) - \rho\vec{V}\cdot grad(e) \qquad (4.1.16)$$

We now make use of the dyadic identity

$$div(\overline{\overline{S}}\cdot\vec{V}) - div(\overline{\overline{S}})\cdot\vec{V} = (\overline{\overline{S}}\cdot grad)\cdot\vec{V} \qquad (4.1.17)$$

Substituting equation (4.1.17) into equation (4.1.16) we obtain equation

$$\rho\frac{\partial e}{\partial t} = (\overline{\overline{S}}\cdot grad)\cdot\vec{V} + div(\overline{\overline{K}}\cdot grad(T))$$

$$+ \frac{\partial Q}{\partial t} + div(q_r) - \rho\vec{V}\cdot grad(e) \qquad (4.1.18)$$

We assume that ρ and the conductivity tensor $\overline{\overline{K}}$ are time independent and that the internal energy source Q and the radiative energy source q_r are both identically zero and apply the local time averaging operator P_{T_p} defined by equation (4.1.5) to all terms of the simplified energy equation (4.1.18) ([33], p 17) to obtain equation

$$\rho\overline{\left(\frac{\partial e}{\partial t}\right)} = \overline{(\overline{\overline{S}}\cdot grad)\cdot\vec{V}} + div(\overline{\overline{K}}\cdot\overline{T}) - \overline{\rho\vec{V}\cdot grad(e)} \qquad (4.1.19)$$

We now use the oscillation theorem which says that if a is smaller than b and if f is continuous on $[a, b]$ then

$$\lim_{n \to \infty} \int_a^b cos(nt)f(t)dt \;=\; 0 \tag{4.1.20}$$

to say that to a good approximation since $\vec{V}$ is a rapidly varying function and e is a slowly varying function we may, in view of (4.1.20), say that to a good approximation,

$$\rho\vec{V} \cdot grad(e) \;=\; 0 \tag{4.1.21}$$

to obtain the first approximate heat equation,

$$\rho\frac{\partial e}{\partial t} \;=\; (\overline{\overline{S}} \cdot grad) \cdot \vec{V} \;+\; div(\overline{\overline{K}} \cdot grad(T)) \tag{4.1.22}$$

We now are prepared to exploit equation (4.1.6) and the relation

$$e \;=\; cT \tag{4.1.23}$$

where e and T are respectively increases in energy density and temperature, and where c is the specific heat to obtain our final form of the heat equation with an elastic energy power density source term. We write for n equal to three or seven,

$$\overline{\overline{S}} \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} S_{(i,j)}\vec{e}_i\vec{e}_j \tag{4.1.24}$$

where

$$\begin{aligned} S_{(i,j)} \;=\; & \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \;+\; \tilde{\mu}\left(\frac{\partial^2 U_i}{\partial x_j \partial t} + \frac{\partial^2 U_j}{\partial x_i \partial t}\right) \\ & \lambda\left(\sum_{k=1}^{n}\frac{\partial U_k}{\partial x_k}\right)\delta_{(i,j)} \;+\; \tilde{\lambda}\left(\sum_{k=1}^{n}\frac{\partial^2 U_k}{\partial x_k \partial t}\right)\delta_{(i,j)} \end{aligned} \tag{4.1.25}$$

We now take the dot product of both sides of equation (4.1.25) with $\vec{V}$ obtaining,

$$\begin{aligned} & \left(\overline{\overline{S}} \cdot grad\right) \cdot \vec{V} \;= \\ & \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} S_{(i,j)}\vec{e}_i\delta_{(j,k)}\frac{\partial}{\partial x_k}\right)\left(\sum_{\ell=1}^{n}\frac{\partial u_\ell}{\partial t}\vec{e}_\ell\right) \\ & \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} S_{(i,j)}\delta_{(i,j)}\delta_{(j,k)}\delta_{(i,\ell)}\frac{\partial^2 U_k}{\partial x_k \partial t}\right) \end{aligned} \tag{4.1.26}$$

We apply the local time average operation defined by (4.1.5) to all terms of equation (4.1.26) making use of (4.1.6) and substitute into equation (4.1.22). Thus, from (4.1.4) we derive the heat equation,

$$\rho c\left(\frac{\partial T}{\partial t}\right) \;-\; div(\overline{\overline{K}} \cdot grad(T)) \;=$$

$$\overline{\sum_{(i,j)\in\mathcal{J}(n)} \bar{\mu}\left(\frac{\partial^2 U_i}{\partial x_j \partial t} + \frac{\partial^2 U_j}{\partial x_i \partial t}\right)^2 + \bar{\lambda}\sum_{i=1}^{n}\left(\frac{\partial^2 U_i}{\partial x_i \partial t}\right)^2} \tag{4.1.27}$$

where the index set is defined by

$$\mathcal{J}(n) = \{(i,j) : j \geq i \text{ and } \{i,j\} \in \{1,2,\cdots,n\}\} \tag{4.1.28}$$

and the internal energy density increase e that appeared in our original energy equation (4.1.4) is related to temperature increase by the relation (4.1.23), where c is the specific heat and T is the temperature increase, which means that since the right side source term of the heat equation (4.1.27) is positive that heat will be generated by vibrations in a Voigt solid.

4.2 Droplet Explosion by Lasers

We now consider energy transfer in a stimulated fluid. Using equation (3.2.3) we define the viscous dissipation function Φ by the rule,

$$\Phi = \mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \right.$$

$$\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2$$

$$\left.\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2\right] \tag{4.2.1}$$

In these terms the energy equation is given by (Anderson, Tannehill, and Pletcher [1], pages 188-189).

$$\frac{\partial}{\partial t}\left\{\rho e + \frac{\rho\vec{v}\cdot\vec{v}}{2}\right\} =$$

$$-div(\rho e\vec{v}) + \vec{f}\cdot\vec{v} +$$

$$div(\overline{\overline{\Pi}}\cdot\vec{v}) - div(\left(\frac{\rho\vec{v}\cdot\vec{v}}{2}\right)\vec{v}) +$$

$$div(\overline{\overline{K}}grad(T)) + \left(\frac{\partial}{\partial t}\right)Q_{in} + \left(\frac{\partial}{\partial t}\right)Q_{out} \tag{4.2.2}$$

We define the enthalpy h as (see Anderson [1], p 188)

$$h = e + \frac{p}{\rho} \tag{4.2.3}$$

where

$e =$ the internal energy including quantum states,
$p =$ the pressure, and
$\rho =$ the density.

To telescope the terms in the energy equation we make use of the vector identity

$$grad(\vec{A}\cdot\vec{B}) = \vec{A}\times curl(\vec{B}) + \vec{B}\times curl(\vec{A}) +$$
$$(\vec{B}\cdot grad)\vec{A} + (\vec{A}\cdot grad)\vec{B} \tag{4.2.4}$$

to observe that

$$\rho\vec{v}\cdot grad\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) =$$
$$\rho\vec{v}\cdot\{\vec{v}\times curl(\vec{v}) + \vec{v}\cdot grad(\vec{v})\} \tag{4.2.5}$$

Interchanging the dot and cross product in equation (4.2.5) we see that since for an arbitrary vector field $\vec{v}$

$$\vec{v}\cdot(\vec{v}\times curl(\vec{v})) = (\vec{v}\times\vec{v})\cdot curl(\vec{v}) = \vec{0} \tag{4.2.6}$$

it follows that

$$\rho\vec{v}\cdot grad\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) = \rho\vec{v}\cdot\{(\vec{v}\cdot grad)(\vec{v})\} \tag{4.2.7}$$

We can then collapse terms in equation (4.2.2) by observing that the momentum equation implies that

$$\vec{v}\cdot\rho(\vec{v}\cdot grad)\vec{v} =$$
$$-\rho\frac{\partial\vec{v}}{\partial t}\cdot\vec{v}$$
$$+\rho\vec{f} - grad(p)\cdot\vec{v} + div(\overline{\overline{\tau}})\cdot\vec{v} \tag{4.2.8}$$

Thus, using equation (1.1.1) and equations (4.2.7) and (4.2.8) the energy equation (4.2.2) may be rewritten in the form,

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} +$$
$$\left(\frac{\partial}{\partial t}\right)Q_{in} + \left(\frac{\partial}{\partial t}\right)Q_{out} +$$
$$\Phi - div(\overline{\overline{K}}grad(T)) \tag{4.2.9}$$

where $(\partial/\partial t)\,Q_{in}$ is given by equation (1.2.4) and Φ is the dissipation function representing the work done by the viscous forces of the fluid. The term representing the transfer by radiation from one part of the fluid to another is given by (Siegel and Howell [51], page 689)

$$\left(\frac{\partial}{\partial t}\right)Q_{out} = div\left(\frac{16\sigma T^3}{3a_R}\cdot grad(T)\right) \tag{4.2.10}$$

This equation may be interpreted as providing a radiation flux across a surface defined by

$$k_r = \frac{16\sigma T^3}{3a_R}, \tag{4.2.11}$$

where a_R is the Rosseland mean absorption coefficient (Siegel [51] , p 504 and Rosseland) and where σ (Siegel [51], page 25) is the hemispherical total emissive power of a black surface into vacuum having a value of

$$\sigma = 5.6696\times10^{-8}\ \ \text{Watts} / (meters^2\ {}^\circ K\ ) \tag{4.2.12}$$

Using equation (4.2.10) and equation (4.2.2) we see that

$$\rho\frac{De}{Dt} = \left(\frac{\partial}{\partial t}\right)Q_{in} + \left(\frac{\partial}{\partial t}\right)Q_{out} +$$

$$(-p\,div(\vec{v})) - div(\overline{\overline{K}}grad(T)) + \Phi \tag{4.2.13}$$

where Φ is the viscous dissipation function given by equation (4.2.1)

4.3 EQUATION OF STATE

In the energy equation (4.2.13) the perfect fluid assumption ([1], p 189) would yield

$$e = c_v T, \tag{4.3.1}$$

where c_v is the specific heat at constant volume, and if we define

$$\gamma = \frac{c_p}{c_v} \tag{4.3.2}$$

where c_p is the specific heat at constant pressure, then the pressure p, the internal energy e and the density ρ are related by ([1], p 189)

$$p = (\gamma - 1)\rho e \tag{4.3.3}$$

5 SUMMARY

Using the definition of velocity (equation 3.1.1) and the equation of state (4.3.3) we see that the number of equations is 5, allowing 3 equations for the three components of the momentum, and while the intial variables are ρ, u, v, w, p, e, and T, we see that since the temperature T is related to e and since pressure is a function of ρ and e, we see that there are now exactly 5 unknowns. This means that locally within the Voigt solid, we can describe the future state of the Voigt solid as a semigroup acting on the conditions at time t_0. If we want to know the value at time t and S is defined so that the solution at time t is given by $S(t - t_0)$ acting on the values at $t = t_0$ of the density ρ, the velocity components u, v, and w, and the temperature T. The semigroup relation,

$$\begin{pmatrix} \rho(t) \\ u(t) \\ v(t) \\ w(t) \\ T(t) \end{pmatrix} = S(t - t_0) \begin{pmatrix} \rho(t_0) \\ u(t_0) \\ v(t_0) \\ w(t_0) \\ T(t_0) \end{pmatrix} \tag{5.0.1}$$

tells us how to get future values of the density ρ, the three velocity components, and the temperature at time t when the values at time t_0 are known.

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Continued Fractions and the Eigenvalues of Spin Weighted Angular Spheroidal Harmonics

D. K. Cohoon

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Contents

1 Introduction

There are at least two approaches to spheroid scattering. These use ordinary spheroidal harmonics and the more general spin weighted angular spheroidal harmonics. The key to both methods is the determination of the eigenvalues of angular spheroidal harmonics. We have proposed a Rayleigh Ritz functional approach, the classical method of estimating eigenvalues for elliptic boundary value problems, and the solution of a transcendental equation involving continued fractions. The latter requires an efficient method of evaluating continued fractions.

2 Rayleigh Ritz Procedures

One method is to use separation of variables to obtain a solution of the scalar Helmholtz equation and then to use the fact the if we multiply the position vector by this single solution and repeatedly apply the curl operation, we only generate a basis for a finite dimensional vector space.

$$\Delta\Psi + k^2\Psi =$$

$$\left\{\frac{\partial}{\partial\xi}\left((\xi^2+1)\frac{\partial\Psi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial\Psi}{\partial\eta}\right)\right.$$
$$\left.+\frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2\Psi}{\partial\phi^2}\right\}$$
$$+\ k^2\frac{d^2}{4}(\xi^2+\eta^2)\Psi\ =\ 0 \tag{2.1}$$

We now seek solutions of equation (2.1) of the form

$$\Psi = R(\xi)S(\eta)exp(im\phi) \tag{2.2}$$

and substitute equation (2 2) into equation (2.1) and then divide all terms of this equation by the function Ψ defined by equation (2.2) after making use of the relationship

$$\frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)} =$$
$$\frac{1}{1-\eta^2}-\frac{1}{\xi^2+1} \tag{2.3}$$

and making the subsitution

$$c^2\ =\ k^2d^2/4 \tag{2.4}$$

we obtain the relation,

$$\left\{\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial}{\partial\eta}S(c,\eta)\right)\right\}/S(c,\eta)$$
$$-\frac{m^2}{1-\eta^2}\ +\ c^2\eta^2\ =$$
$$-\ \left\{\frac{\partial}{\partial\xi}\left((\xi^2+1)\frac{\partial R(c,\xi)}{\partial\xi}\right)\right\}/R(c,\xi)\ +$$
$$\frac{m^2}{\xi^2+1}\ +\ c^2\xi^2\ =\ -\ \lambda_{(m,n)} \tag{2.5}$$

From equation (2.5) we obtain a kind of Rayleigh Ritz functional for the value of $\lambda_{(m,n)}$. Equation (2.5) tells us that

$$\lambda_{(m,n)}\ =$$
$$\left\{\int_{1-}^{1}\left[(1-\eta^2)\left(\frac{dS}{d\eta}\right)^2+S^2\cdot\left\{(-c^2\eta^2)+\frac{m^2}{1-\eta^2}\right\}\right]d\eta\right\}/\left\{\int_{-1}^{1}S^2d\eta\right\} \tag{2.6}$$

We note that when c is equal to zero, we are dealing with a sphere and that the angular functions are the associated Legendre functions $P_n^m(\eta)$ so it makes sense that we want S to behave like the function $P_n^m(\eta)$ when c is zero. We note that either $n-m$ is even or odd, and we know the initial conditions exactly in each case. We use partial derivative notation for functions $G(c,\eta)$ and note that

$$D_2G(c,0) = \underset{\eta\to 0}{LIM}\ \frac{\partial G}{\partial\eta} \tag{2.7}$$

and define the intial conditions for the second order ordinary differential equation satisfied by the functions $S(c,\eta)$. We find that if $n-m$ is an even integer

$$S_{(m,n)}(c,0) = \left\{(-1)^{(n-m)/2}(n+m)!\right\} / \left\{2^n\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!\right\} \tag{2.8}$$

and

$$D_2S_{(m,n)}(c,0) = 0 \tag{2.9}$$

and when $n-m$ is an odd number that

$$S_{(m,n)}(c,0) = 0 \tag{2.10}$$

and that

$$D_2S_{(m,n)}(c,0) = \left\{(-1)^{(n-m-1)/2}(n+m+1)!\right\} / \left\{2^n\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!\right\} \tag{2.11}$$

With these initial conditions we have completely specified S and its partial derivative and mixed partial derivative as a function of η, c, and λ and we also know that

$$\lambda(0) = n(n+1) \tag{2.12}$$

This gives us an initial value problem and an ordinary differential equation

$$\lambda'(c) = F(c,\lambda) \tag{2.13}$$

where the function F is determined by differentiating both sides of equation (2.6) with respect to c and collecting terms involving $\lambda'(c)$, and then dividing all terms by the coefficient of $\lambda'(c)$ to get the first order ordinary differential equation (2.13). By the uniqueness of the Cauchy problem, different initial values cannot lead to the same eigenvalue at

$$c = k\cdot\frac{d}{2} \tag{2.14}$$

This is effective if c is real, but if k is complex, then we think of c as being a function of a paramter s defined by

$$c(s) = s\cdot k\cdot\frac{d}{2} \tag{2.15}$$

and with the same initial condition develop an ordinary differential equation of the form,

$$\lambda'(s) = G(s,\lambda) \tag{2.16}$$

We can also use continued fraction relationships to get the values of λ Separation of variables applied to the scalar Helmholtz equation in spheroidal coordinates yields the ordinary differential equation

$$\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial}{\partial\eta}S(c,\eta)\right) + \left(\lambda + c^2\eta^2 - \frac{m^2}{1-\eta^2}\right)S(c,\eta) = 0 \tag{2.17}$$

We seek a solution, $S(c,\eta)$ which is bounded at η equal to plus and minus one only for a discrete set of eigenvalues λ. We obtain these solutions as one of two odd or even power series,

$$S^{(o)}(\eta) \;=\; \eta\left(1-\eta^2\right)^{m/2}\left[\sum_{k=0}^{\infty} C_{2k}\left(1-\eta^2\right)\right] \tag{2.18}$$

or

$$S^{(e)}(\eta) \;=\; \left(1-\eta^2\right)^{m/2}\left[\sum_{k=0}^{\infty} C_{2k}\left(1-\eta^2\right)^k\right] \tag{2.19}$$

We take derivatives with respect to η of the power series defined by equation (2.18) and when necessary make repeated use of the tautology

$$-A\cdot\eta^2 \;=\; (+A)\cdot(1\;-\;\eta^2)\;-\;A \tag{2.20}$$

we deduce after collecting terms that

$$\frac{d}{d\eta}S^{o}(\eta) \;=\; \left(1-\eta^2\right)^{m/2}\sum_{k=0}^{\infty} C_{2k}\left[1+m+2\cdot k\right]\left(1-\eta^2\right)^k$$

$$\left(1-\eta^2\right)^{m/2}\sum_{k=0}^{\infty} C_{2k}\left[-m-2k\right]\left(1-\eta^2\right)^{k-1} \tag{2.21}$$

while

$$\frac{d}{d\eta}S^{e}(\eta) \;=\; \eta\left(1-\eta^2\right)^{m/2}\sum_{k=0}^{\infty} C_{2k}\left[-m-2k\right]\left(1-\eta^2\right)^{k-1} \tag{2.22}$$

If we substitute in the power series, we get a seemingly infinite set of recursion relations; a closer examination reveals that we can use continued fractions to eliminate the a priori unknown coefficients C_{2k} and get a single parameterized continued fraction expression for λ of the form

$$F(\lambda(s),n,m,c(s)) \;=\; 0 \tag{2.23}$$

where if

$$c(0) \;=\; 0 \tag{2.24}$$

the equation is that of the associated Legendre function $P_n^m(\eta)$ which means that

$$\lambda(0) \;=\; n\cdot(n+1) \tag{2.25}$$

Thus, the eigenvalues can be systematically determined, since $c(s)$ could be written as s times the actual value proportional to the distance between focal points of the spheroid, as the solution to the initial value problem

$$\lambda'(s) \;=\; \frac{D_4F(\lambda,n,m,c)c'(s)}{D_1F(\lambda,n,m,c)} \tag{2.26}$$

We simply solve the ordinary differential equation (2.26) to get to the eigenvalue, which then, because of the original recursion relationships gives us all the values of C_{2k}; the spheroidal harmonics are systematically determined, even when the material properties are complex. The spin weighted angular spheroidal harmonics can be determined by a similar method.

A generalization of (2.17) is the ordinary differential equation for the spin weighted angular spheroidal functions ([6]) is given by

$$\frac{\partial}{\partial \eta}\left((1-\eta^2)\frac{\partial}{\partial \eta}S(c,\eta)\right) +$$

$$\left(\lambda + \gamma^2 \cdot \eta^2 - \frac{m^2 + s^3}{1-\eta^2} - \frac{2 \cdot m \cdot s \cdot \eta}{1-\eta^2} - 2 \cdot \gamma \cdot s \cdot \eta\right) S(c,\eta) = 0 \tag{2.27}$$

We note that we get the usual differential equation (2.17) simply by setting s equal to zero in equation (2.27). The method of using the spin weighted angular spheroidal harmonics to calculate scattering by spheroids is discussed in ([10]). The key to success is the determination of the eigenvalues. We attempt to find solutions of equation (2.17) of the form of equation (2.18) or equation (2.19). It appears at first glance that there would be a term involving the reciprocal of $(1-\eta^2)$, but this term exactly cancels out, which means that if we can simply solve the resulting recursion relationship for some values of λ, we have our bounded solution. The values of λ for which we can find bounded solutions of equation (2.17) are eigenvalues and are determined by solving a transcendental equation in λ involving a continued fraction. A similar but more complex situation arises in determining the eigenvalues associated with the spin weighted angular spheroidal harmonics satisfying the more general equation (2.27).

The precise solution of a spheroid scattering problem will provide a convincing benchmark for the general surface or volume integral equation method.

3 Continued Fractions

The following theorems give us a practical means for evaluating continued fractions on a digital computer with a minimum of round off error.

Theorem 3.1 *Let $b_0, b_1, b_2, \cdots$ and $a_0, a_1, a_2, \cdots$ be two sequences of complex numbers. Assume that W_0 is equal to b_0 and let*

$$W_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots + \cfrac{a_n}{b_n}}}} \tag{3.1}$$

for all integers n greater than zero. Define initial values of a recursion by the relation,

$$(A_{-1}, B_{-1}, A_0, B_0) = (1, 0, b_0, 1) \tag{3.2}$$

Then define A_n and B_n for integers n larger than zero by the recursion

$$A_n = b_n A_{n-1} + a_n A_{n-2} \tag{3.3}$$

and the relation

$$B_n = b_n B_{n-1} + a_n B_{n-2} \tag{3.4}$$

Assume that b_n is nonzero and that for every positive integer n that

$$b_{n-1} \cdot b_n + a_n \neq 0 \tag{3.5}$$

for all positive integers n. Then for all positive integers n we have

$$W_n = \frac{A_n}{B_n} \tag{3.6}$$

Proof: We proceed by induction on n. The statement (3.6) which we shall call proposition $\mathcal{P}(n)$ is true trivially if n is equal to zero and also for n equal to 1, since the initial conditions (3.2) and the recursion relations (3.3) and (3.4) imply that

$$\frac{A_1}{B_1} = \frac{b_1 \cdot b_0 + a_1 \cdot 1}{b_1 \cdot 1 + a_1 \cdot 0} = b_0 + \frac{a_1}{b_1} = W_1 \tag{3.7}$$

Thus, assume that n is larger than one and that if m is a positive integer that is strictly smaller than m, then propositon $\mathcal{P}(m)$ or the statement,

$$W_m = \frac{A_m}{B_m}, \tag{3.8}$$

is true. We now define a shorter continued fraction related to the W_n, which is defined by equation (3.1) by the rule,

$$W_n = W_n^*$$

where

$$W_n^* = b_0^* + \cfrac{a_1}{b_1^* + \cfrac{a_2^*}{b_2^* + \cfrac{a_3^*}{b_3^* + \cdots + \cfrac{a_{n-1}^*}{b_{n-1}^*}}}} \tag{3.9}$$

where

$$b_{n-1}^* = b_{n-1} \cdot b_n + a_n, \tag{3.10}$$

$$a_{n-1}^* = a_{n-1} \cdot b_n, \tag{3.11}$$

and

$$a_j^* = a_j \tag{3.12}$$

and

$$b_j^* = b_j \tag{3.13}$$

for all j satisfying

$$j \in \{0, 1, \cdots, n-2\} \tag{3.14}$$

Defining, as before,

$$A_{n-1}^* = b_{n-1} \cdot A_{n-2}^* + a_{n-3} \cdot A_{n-3}^* \tag{3.15}$$

and the relation

$$B_{n-1}^* = b_n \cdot B_{n-2}^* + a_n \cdot B_{n-3}^* \tag{3.16}$$

and realizing that

$$A_{n-2}^* = A_{n-2}, \tag{3.17}$$

$$A_{n-3}^* = A_{n-3}, \tag{3.18}$$

$$B_{n-2}^* = B_{n-2}, \tag{3.19}$$

and

$$B_{n-3}^* = B_{n-3}, \tag{3.20}$$

we see that by the inductive hypothesis

$$W_n = W_{n-1}^* = \frac{A_{n-1}^*}{B_{n-1}^*} =$$

$$\frac{b_{n-1}^* \cdot A_{n-2} + a_{n-1}^* \cdot A_{n-3}}{b_{n-1}^* \cdot B_{n-2} + a_{n-1}^* \cdot B_{n-3}} \tag{3.21}$$

Substituting (3.10) and(3.11) into equation (3.21) and collecting coefficients of a_n and b_n we see that

$$W_n = \frac{(b_{n-1} \cdot b_n + a_n) \cdot A_{n-2} + (a_{n-1} \cdot b_n) \cdot A_{n-3}}{(b_{n-1} \cdot b_n + a_n) \cdot B_{n-2} + (a_{n-1} \cdot b_n) \cdot B_{n-3}} =$$

$$\frac{b_n \cdot (b_{n-1} \cdot A_{n-2} + a_{n-1} \cdot A_{n-3}) + a_n \cdot A_{n-2}}{b_n \cdot (b_{n-1} \cdot B_{n-2} + a_{n-1} \cdot B_{n-3}) + a_n \cdot B_{n-2}} \tag{3.22}$$

Using the recursion relations (3.3) and (3.4) for A_n and B_n and substituting these into equation (3.22) we see that

$$W_n = \frac{b_n \cdot A_{n-1} + a_n \cdot A_{n-2}}{b_n \cdot B_{n-1} + a_n \cdot B_{n-2}} = \frac{A_n}{B_n} \tag{3.23}$$

In view of equation (3.23) and our original definition (3.1) the theorem is proven by induction on n.

With any fractional representation, one is always concerned about division by zero. If we regard the ground field to be the quotient field of the integral domain of functions which are holomorphic on some open set Ω of the field of complex numbers $\mathbf{C}$, then if each a_n is a constant function, and if each $b_n \cdot b_{n-1}$ and b_0 is nonzero, then b_1 and $b_{n-1} \cdot b_n + a_n$ are nonzero meromorphic functions for all nonnegative integers, which means that under these hypotheses, all operations of the partial continued fractions W_n are defined for all but at most a countable collection of complex numbers.

The next theorem shows us how to eliminate some of the variables in the continued fractions.

Theorem 3.3 *Let us suppose that a continued fraction*

$$W_n = \frac{A_n}{B_n} \tag{3.24}$$

is defined by the initial conditions

$$(A_{-1}, A_0, B_{-1}, B_0) = (1, b_0, 0, 1) \tag{3.25}$$

and the recursion relations

$$A_n = b_n \cdot A_{n-1} + A_{n-2} \tag{3.26}$$

and

$$B_n = b_n \cdot B_{n-1} + B_{n-2} \tag{3.27}$$

where b_n and $b_n \cdot b_{n-1} + 1$ are nonzero for all nonnegative integers n. Then if for all nonnegative n we introduce a symbolic representation of a continued fraction by the relation

$$b_0 + \frac{1}{[b_1, b_2, \cdots, b_n]} =$$

$$[b_0, b_1, \cdots, b_n] = W_n \tag{3.28}$$

then the A_n and B_n may be represented in terms of these continued fractions by the relations,

$$A_n = [b_0] \cdot [b_1, b_0] \cdot [b_2, b_1, b_0] \cdots [b_n, b_{n-1}, \cdots, b_1, b_0] \tag{3.29}$$

for all nonnegative integers n and

$$B_n = [b_1] \cdot [b_2, b_1] \cdot [b_3, b_2, b_1] \cdots [b_n, b_{n-1}, \cdots, b_2, b_1] \tag{3.30}$$

for all positive integers n

Proof of Theorem. We proceed by double induction on n. Let $\mathcal{P}(n)$ be the assertion that equation (3.29) is valid and let $\mathcal{Q}(n)$ be the assertion that equation (3.30) is valid, where n is a positive integer. Now if we set

$$A_{-2} = 0 \tag{3.31}$$

then $\mathcal{P}(0)$ is the statement

$$A_0 = [b_0] = \frac{A_0}{B_0} = \frac{b_0}{1} \tag{3.32}$$

follows simply from the initial conditions (3.25) and is also consistent with the recursion relation (3.26) which has the form,

$$A_0 = b_0 \cdot A_{0-1} + A_{0-2} \tag{3.33}$$

The statement $\mathcal{P}(1)$ is valid since

$$[b_0] \cdot [b_1, b_0] = b_0 \cdot \left(b_1 + \frac{1}{b_0}\right)$$

$$= b_0 \cdot b_1 + 1 = b_1 \cdot A_0 + A_{-1} \tag{3.34}$$

in view of the initial conditions (3.25) and the recursion relation (3.26). This proves the validity of $\mathcal{P}(1)$. Now assume that $\mathcal{P}(m)$ is valid for m not exceeding n and attempt to prove $\mathcal{P}(n+1)$. We use the recursion relations (3.26) to define

$$A_{n+1} = b_{n+1} \cdot A_n \; + \; A_{n-1} \tag{3.35}$$

It follows from the inductive hypothesis and substituting the symbol product representation (3.29) of A_n and A_{n-1} into equation (3.35) that

$$\begin{aligned} A_{n+1} \;=\; & b_{n+1} \cdot ([b_0] \cdot [b_1, b_0] \cdot [b_2, b_1, b_0] \cdots [b_n, b_{n-1}, \cdots, b_1, b_0]) \\ & + \; ([b_0] \cdot [b_1, b_0] \cdot [b_2, b_1, b_0] \cdots [b_{n-1}, b_{n-2}, \cdots, b_1, b_0]) \end{aligned} \tag{3.36}$$

Dividing both sides of equation (3.36) by the same quantity, A_{n-1}, we see that

$$\begin{aligned} & \left(\frac{A_{n+1}}{[b_0] \cdot [b_1, b_0] \cdot [b_2, b_1, b_0] \cdots [b_{n-1}, b_{n-2}, \cdots, b_1, b_0]} \right) \;= \\ & [b_n, b_{n-1}, \cdots, b_1, b_0] \cdot \left(b_{n+1} \; + \; \frac{1}{[b_n, b_{n-1}, \cdots, b_1, b_0]} \right) \end{aligned} \tag{3.37}$$

But by the definition of continued fraction

$$\begin{aligned} & [b_{n+1}, b_n, b_{n-1}, \cdots, b_1, b_0] \;= \\ & \left(b_{n+1} \; + \; \frac{1}{[b_n, b_{n-1}, \cdots, b_1, b_0]} \right) \end{aligned} \tag{3.38}$$

Substituting equation (3.38) into equation (3.37) and multiplying both sides of this rewritten equation by the symbol product in the denominator of (3.37) we see that

$$\begin{aligned} & A_{n+1} \;= \\ & \{[b_0] \cdot [b_1, b_0] \cdot [b_2, b_1, b_0] \cdots [b_{n-1}, b_{n-2}, \cdots, b_1, b_0]\} \cdot [b_n, b_{n-1}, \cdots, b_1, b_0]([b_{n+1}, b_n, \cdots, b_1, b_0]) \end{aligned} \tag{3.39}$$

This shows that $\mathcal{P}(n+1)$ is a consequence of $\mathcal{P}(n)$.

We now proceed to prove the validity of $\mathcal{Q}(n)$, or the assertion that (3.30) is valid for the postive integer n. Observe that even $\mathcal{Q}(0)$ is true if we assume that a product of an empty set of integers is 1 and the statement $\mathcal{Q}(1)$ is true since the recursion relation (3.27) says that

$$B_1 \;=\; b_1 \cdot B_0 \; + \; B_{-1} \tag{3.40}$$

which in view of the initial conditions (3.25) implies that

$$B_1 \;=\; b_1 \;=\; [b_1] \tag{3.41}$$

which proves that $\mathcal{Q}(1)$ is true. Now assume that $\mathcal{Q}(m)$ is true for all m not exceeding n and attempt to prove that $\mathcal{Q}(n+1)$ is valid. Note that the recursive definition (3.27) implies that

$$B_{n+1} \;=\; b_{n+1} \cdot B_n \; + \; B_{n-1} \tag{3.42}$$

and that by the inductive hypothesis we can substitute equations (3.30) and into equation (3.42) obtaining

$$B_{n+1} = b_{n+1} \cdot ([b_1] \cdot [b_2, b_1] \cdot [b_3, b_2, b_1] \cdots [b_{n-1}, b_{n-2}, \cdots, b_2, b_1][b_n, b_{n-1}, \cdots, b_2, b_1]) + ([b_1] \cdot [b_2, b_1] \cdot [b_3, b_2, b_1] \cdots [b_{n-1}, b_{n-2}, \cdots, b_2, b_1]) \tag{3.43}$$

Dividing both sides of equation (3.43) by the product of the first $n-1$ symbols we have

$$\left(\frac{B_{n+1}}{[b_1] \cdot [b_2, b_1] \cdot [b_3, b_2, b_1] \cdots [b_{n-1}, b_{n-2}, \cdots, b_2, b_1]}\right) = b_{n+1}([b_n, b_{n-1}, \cdots, b_2, b_1]) + 1 = [b_n, b_{n-1}, \cdots, b_2, b_1]\left(b_{n+1} + \frac{1}{[b_n, b_{n-1}, \cdots, b_2, b_1]}\right) \tag{3.44}$$

Using the interpretation of the continued fraction in definition (3.28) which says that

$$[b_{n+1}, b_n, \cdots, b_2, b_1] = b_{n+1} + \left(\frac{1}{[b_n, b_{n-1}, \cdots, b_2, b_1]}\right) \tag{3.45}$$

Substituting (3.45) into (3.44) we see that

$$B_{n+1} = [b_1] \cdot [b_2, b_1] \cdot [b_3, b_2, b_1] \cdots [b_n, b_{n-1}, \cdots, b_2, b_1][b_{n+1}, b_n, \cdots, b_2, b_1] \tag{3.46}$$

which completes the proof of the validity of $Q(n)$ for all positive integers n. This completes the proof of the theorem.

We can use this to stabilize the numerical computation of any continued fraction by making transformations which reduce the continued fraction to the form where each a_n is equal to one by introducing new variables.

Theorem 3.3 *Under the hypothesis that each b_n and a_n is nonzero and*

$$b_{n-1} \cdot b_n + a_n \neq 0 \tag{3.47}$$

if we introduce the new variables

$$\tilde{b}_0 = b_0 \tag{3.48}$$

and

$$\tilde{b}_1 = \frac{b_1}{a_1} \tag{3.49}$$

and

$$\tilde{b}_{2n} = \left(\frac{\prod_{k=1}^{n} a_{2k-1}}{\prod_{k=1}^{n} a_{2k}}\right) \cdot b_{2n} \tag{3.50}$$

and if

$$\tilde{b}_{2n+1} = \left(\frac{\prod_{k=1}^{n} a_{2k}}{\prod_{k=1}^{n} a_{2k-1}}\right) \cdot \left\{\frac{b_{2n+1}}{a_{2n+1}}\right\} \tag{3.51}$$

then if W_n is the general continued fraction defined by (3.1) it follows that

$$W_n = [\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n] \tag{3.52}$$

Proof of Theorem. We proceed by induction on n. Let $\mathcal{P}(n)$ be the statement that if we define W_n by (3.1) then (3.52) is valid, where the $\tilde{b}_n$ are defined by equations (3.50) and (3.51). The validity of the assertion $\mathcal{P}(1)$ is simply the statement that

$$W_1 = \tilde{b}_0 + \frac{1}{\tilde{b}_1} =$$

$$b_0 + \frac{1}{b_1/a_1} = b_0 + \frac{a_1}{b_1} \tag{3.53}$$

which is exactly equation (3.1) for n equal to 1. The assertion that $\mathcal{P}(2)$ is valid is, since the definition, (3.50), implies that

$$\tilde{b}_2 = \left(\frac{a_1}{a_2}\right) \cdot b_2 \tag{3.54}$$

equivalent to

$$W_2 = \tilde{b}_0 + \frac{1}{[\tilde{b}_1, \tilde{b}_2]} = \tilde{b}_0 + \frac{1}{(\tilde{b}_1 + 1/\tilde{b}_2)} =$$

$$b_0 + \left(\frac{a_1}{a_1}\right) \cdot \frac{1}{\{(b_1/a_1) + 1/((a_1/a_2) \cdot b_2)\}} = b_0 + \frac{a_1}{b_1 + (a_2/b_2)} \tag{3.55}$$

which in view of equation (3.1) is true. We now assume that $\mathcal{P}(m)$ is true for m less than or equal to n and then attempt to prove the validity of $\mathcal{P}(n)$, thereby completing the proof of the theorem by induction. We introduce the transformed variables $\tilde{b}_{2k}^{(t)}$ and $\tilde{b}_{2k+1}^{(t)}$ that will help us define the tail of the continued fraction that are defined by the rules,

$$\tilde{b}_{2k}^{(t)} = \frac{\tilde{b}_{2k}}{a_1} \tag{3.56}$$

and

$$\tilde{b}_{2k+1}^{(t)} = \tilde{b}_{2k+1} \cdot a_1 \tag{3.57}$$

We also introduce, for this theorem, the shifted sequence variables

$$a_n^{(s)} = a_{n+1} \tag{3.58}$$

and

$$b_n^{(s)} = b_{n+1} \tag{3.59}$$

It will then follow from the inductive hypothesis that

$$b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \cdots + \cfrac{a_n}{b_n}}}}$$

$$= \left[\bar{b}_1^{(t)}, \bar{b}_2^{(t)}, \cdots, \bar{b}_n^{(t)}\right] = \left[\bar{b}_0^{(s)}, \bar{b}_1^{(s)}, \cdots, \bar{b}_{n-1}^{(s)}\right] \tag{3.60}$$

Consequently, by $\mathcal{P}(2)$, which we have proven it follows that

$$\left[\bar{b}_0^{(t)}, \bar{b}_1^{(t)}, \bar{b}_2^{(t)}, \cdots, \bar{b}_n^{(t)}\right] =$$

$$b_0 + \frac{a_1}{\left[\bar{b}_1^{(t)}, \bar{b}_2^{(t)}, \cdots, \bar{b}_n^{(t)}\right]} =$$

$$b_0 + \frac{1}{(1/a_1) \cdot \left[\bar{b}_1^{(t)}, \bar{b}_2^{(t)}, \cdots, \bar{b}_n^{(t)}\right]} \tag{3.61}$$

To complete the proof of the theorem we make use of the following lemma.

Lemma 3.1 *We let* $[c_0, c_1, \cdots c_n]$ *be the continued fraction defined by*

$$[c_0, c_1, \cdots c_n] = c_0 + \frac{1}{[c_1, c_2, \cdots c_n]} \tag{3.62}$$

Then for all nonzero constants α *we have*

$$\alpha \cdot [c_0, c_1, \cdots c_{2n}] =$$

$$[(\alpha \cdot c_0), (c_1/\alpha), (\alpha \cdot c_2) \cdots \alpha \cdot c_{2n}] \tag{3.63}$$

and

$$\alpha \cdot [c_0, c_1, \cdots c_{2n+1}] =$$

$$[\alpha \cdot c_0, (c_1/\alpha), (\alpha \cdot c_2) \cdots (c_{2n+1}/\alpha)] \tag{3.64}$$

The lemma will be proven by induction on n. We prove (3.63) by observing that by definition,

$$\alpha \cdot [c_0, c_1, \cdots c_{2n}]$$

$$\alpha \cdot c_0 + \frac{\alpha}{[c_1, \cdots c_{2n}]} \tag{3.65}$$

We can then use the inductive hypothesis to show that

$$\alpha \cdot c_0 + \frac{1}{(1/\alpha) \cdot [c_1, \cdots c_{2n}]} =$$

$$\alpha \cdot c_0 + \frac{1}{[c_1/\alpha, \alpha c_2, c_3/\alpha \cdots \alpha c_{2n}]} =$$

$$[\alpha c_0, c_1/\alpha, \alpha c_2, c_3/\alpha, \ldots, \alpha c_{2n}] \tag{3.66}$$

which completes the proof of equation (3.63) by induction on n.

Next we use induction to establish equation (3.64) by assuming that n were equal to $2 \cdot m + 1$. Then by definition

$$\alpha \cdot [c_0, c_1, c_2, \cdots c_{2m+1}] =$$

$$\alpha \cdot c_0 + \frac{1}{(1/\alpha) \cdot [c_1, c_2, \cdots c_{2n+1}]} \tag{3.67}$$

and by the inductive hypothesis we conclude that equation (3.67) implies that

$$\alpha \cdot [c_0, c_1, c_2, \cdots c_{2m+1}] =$$

$$\alpha \cdot c_0 + \frac{1}{[c_1/\alpha, \alpha \cdot c_2, \cdots c_{2m+1}/\alpha]} \tag{3.68}$$

Applying the definition (3.62) to equation (3.68) we conclude that

$$\alpha \cdot [c_0, c_1, c_2, \cdots c_{2m+1}] =$$

$$[\alpha \cdot c_0, c_1/\alpha, \alpha \cdot c_2, \cdots c_{2m+1}/\alpha] \tag{3.69}$$

which proves equation (3.64) and completes the proof of the Lemma.

By applying the lemma to equation (3.61) we see that

$$b_0 + \frac{a_1}{\left[\tilde{b}_1^{(i)}, \tilde{b}_2^{(i)}, \cdots, \tilde{b}_n^{(i)}\right]} =$$

$$b_0 + \frac{1}{\left[\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n\right]} \tag{3.70}$$

in view of equations (3.57) and (3.56). From equation (3.70) and (3.48) and it follows that (3.52) and the theorem are valid.

We can use these theorems to compute values of convergent continued fractions by numerically stable algorithms.

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An Algorithm for the Eigenvalues of the Angular Spheroidal Harmonics and An Exact Solution to the Problem of Describing Electromagnetic Interaction with Anisotropic Structures Delimited by N Confocal Spheroids

D. K. Cohoon

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Contents

1 Introduction

Prolate spheroids are cigars and footballs and oblate spheroids are falling raindrops and doorknobs. A spheroid is an ellipse rotated about an axis. If it is rotated about a major axis it is a prolate spheroid. If it is rotated about a minor axis, it is an oblate spheroid. In the halls of Congress a certain young representative had his desk in a most undesirable location; for some reason, however, he was able to rise instantly and give brilliant rebuttals of the arguments of his opposition. It turned out that the roof was a spheroid and his desk was at one of the focal points and the desk of the opposition was at the other focal point. He could hear the whispered planning of the oppositon long before they got up to speak. Unlike the wedding guest described 2000 years ago, he refused to move up to a place of greater honor, and, his secret remaining with himself, others were content to allow him to remain in his more humble post.

Spheroid scattering is important because it provides challenges for general purpose codes, and because one is interested in the propagation of electromagnetic information through clouds of spheroids, such as falling raindrops. The computer codes developed may also have a bearing on the design of liquid crystal devices, such as liquid crystal television sets and computer monitors which would, as they use natural room light, be far safer for the users, often young girls, than cathode ray tube (CRT) devices currently in use. Young children, in poor urban settings, often spend hours huddled close to television sets. If they are going to do this anyway, let us, for the sake of the children, make television screens safer with a liquid crystal design. The ability to remember sight together with sound, may provide a way to teach and make literate a larger segment of human society all over the world; we have many serious problems to solve, and no one knows from where the genius to create a solution may come.

The Helmholtz equation can be solved in spheroidal coordinates, and using this solution, we can obtain solutions of the Faraday and Ampere Maxwell equations. Note that if Ψ is a solution of the Helmholtz equation, then if $\vec{r}$ is the radial vector, then

$$\vec{M} = curl(\vec{r} \cdot \Psi) \tag{1.1}$$

and

$$\vec{N} = (1/k)curl(\vec{M}) \tag{1.2}$$

can be used to obtain a solution of Maxwell's equations. We proceed to define these computations in spheroidal coordinates.

2 Spheroidal Coordinates

Consider an ellipse with foci at $(0, -d/2)$ and $(0, d/2)$ on the z axis and if

$$r^2 = x^2 + y^2 \tag{2.1}$$

and (r, z) is a point on the generating curve for the spheroid, then if we define for r_1 being the distance between (r, z) and $(0, -d/2)$ and if r_2 is the distance between $(0, d/2)$ and (r, z) then if we define $\tilde{\xi}$ by the rule,

$$\tilde{\xi} = (r_1 + r_2)/(2 \cdot c) \tag{2.2}$$

and define $\tilde{\eta}$ by the relation,

$$\tilde{\eta} = (r_1 - r_2)/(2 \cdot c) \tag{2.3}$$

where c is a constant. We shall actually use a slightly different set of coordinates that are qualitatively the same We can define points on the surface of the spheroid as all these points (ξ, η, ϕ) for which ξ is a constant, which since an ellipse is the locus of points such that the sum of the distances from fixed foci is a constant is embodied in the definition of ξ given by equation, (2.2). The other coordinate surface defined by setting η equal to a constant and ϕ equal to a constant is a hyperbola, as this says simply that the difference of the distances between two foci is a constant. We give an alternative definition of the spheroidal coordinates and show that this definition is compatible with the more intuitive definitions

of equations (2.2) and (2.3) The relations between spheroidal and Cartesian coordinates are given by

$$x = \frac{d}{2}\left[(1-\eta^2)(\xi^2+1)\right]^{1/2} cos(\phi) \tag{2.4}$$

and

$$y = \frac{d}{2}\left[(1-\eta^2)(\xi^2+1)\right]^{1/2} sin(\phi) \tag{2.5}$$

and

$$z = \frac{d}{2}\eta\xi \tag{2.6}$$

Going back to the equation for an oblate spheroid we have that

$$\begin{gathered}(x^2+y^2)/A^2 + z^2/B^2 = \\ \frac{(d^2/4)(1-\eta^2)(\xi^2+1)}{A^2} + \frac{(d^2/4)\eta^2\xi^2}{B^2} = \\ \frac{d^2}{4}\left[\frac{1-\eta^2}{(d/2)^2} + \frac{\eta^2}{(d/2)^2}\right] = 1\end{gathered} \tag{2.7}$$

if we simply let A and B be defined by

$$A = \frac{d}{2}\sqrt{\xi^2+1} \tag{2.8}$$

and

$$B = \frac{d}{2} \mid \xi \mid \tag{2.9}$$

For the oblate spheroid, we have

$$A > B \tag{2.10}$$

and the foci of the ellipse may be thought to be on the x axis located at

$$x = C = \sqrt{A^2 - B^2} = d/2 \tag{2.11}$$

and the sum of the distances from a fixed point on the surface to the two foci is $2A$ which happens to be

$$2A = d\sqrt{\xi^2+1} = r_1 + r_2 \tag{2.12}$$

If we compare equation (2.12) with the earlier equation (2.2) we can see easily the connection between ξ and $\tilde{\xi}$ and that setting either one of these equal to a constant defines a surface of a spheroid.

We now try to develop the unit vectors in the direction of the normals to the coordinate surfaces $\xi = constant$ or $\eta = constant$. Note that if we had a general coordinate transformation relationship

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{pmatrix} \tag{2.13}$$

and the unit vector in the direction of the normal to the coordinate surface

$$u = constant \tag{2.14}$$

is given by

$$\vec{e}_u = \frac{d\vec{R}}{du} / \left(\| \frac{d\vec{R}}{du} \| \right) \tag{2.15}$$

where :

$$\vec{R} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \tag{2.16}$$

If we imagine an arc in three dimensional space and try to describe it in Cartesian and spheroidal coordinate. Assume that the arc $\vec{R}(t)$ is defined as an orbit defined by a continuous parameter t. Let $s(t_2)$ minus $s(t_1)$ denote the arc length between $\vec{R}(t_2)$ and $\vec{R}(t_1)$ on this curve so that

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \tag{2.17}$$

In order to get values of parameters h_ξ, h_η, and h_ϕ so that we may express the Laplacian and curl operations in spheroidal coordinates we observe that equation (2.4) implies that

$$\frac{\partial x}{\partial \xi} = \frac{d}{2}\left[(1-\eta^2)\right]^{1/2} \xi \left[(\xi^2+1)\right]^{-1/2} cos(\phi) \tag{2.18}$$

From equation (2.5) we see that

$$\frac{\partial y}{\partial \xi} = \frac{d}{2}\left[(1-\eta^2)\right]^{1/2} \xi \left[(\xi^2+1)\right]^{-1/2} sin(\phi) \tag{2.19}$$

From equation (2.6) we see that

$$\frac{\partial z}{\partial \xi} = \frac{d}{2}\eta \tag{2.20}$$

Thus, using the unit vector equation (2.15) and equations (2.18) and (2.19) and (2.20) we see that the unit vector $\vec{e}_\xi$ is given by

$$\begin{aligned} \vec{e}_\xi = & \sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}\left[\xi\sqrt{\frac{1-\eta^2}{\xi^2+1}}cos(\phi)\vec{e}_x + \right. \\ & \left. \xi\sqrt{\frac{1-\eta^2}{\xi^2+1}}sin(\phi)\vec{e}_y + \eta\vec{e}_z\right] \end{aligned} \tag{2.21}$$

Thus, we see that the length factors In an analogous manner we write down the unit vector $\vec{e}_\eta$ by the rule

$$\begin{aligned} \vec{e}_\eta = & \sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}\left[-\eta\sqrt{\frac{\xi^2+1}{1-\eta^2}}cos(\phi)\vec{e}_x + \right. \\ & \left. -\eta\sqrt{\frac{\xi^2+1}{1-\eta^2}}sin(\phi)\vec{e}_y + \xi\vec{e}_z\right] \end{aligned} \tag{2.22}$$

We observe from equations (2.4), (2.5), and (2.6) that

$$\frac{\partial x}{\partial \phi} = -\frac{d}{2}\left[(1-\eta^2)(\xi^2+1)\right]^{1/2} sin(\phi), \tag{2.23}$$

$$\frac{\partial y}{\partial \phi} = \frac{d}{2}\left[(1-\eta^2)(\xi^2+1)\right]^{1/2} cos(\phi) \tag{2.24}$$

and

$$\frac{\partial z}{\partial \phi} = 0 \tag{2.25}$$

Finally, again making use of the equation (2.15) and equations (2.23) and (2.24) and (2.25) we see that the unit vector $\vec{e}_\phi$ is given by

$$\vec{e}_\phi = -sin(\phi)\vec{e}_x + cos(\phi)\vec{e}_y \tag{2.26}$$

It is clear from the definition, equation (2.15) used in creating equations (2.21), (2.22), and (2.23) that there are scalar functions h_ξ, h_η, and h_ϕ of ξ and η that satisfy

$$h_\xi \vec{e}_\xi = \frac{\partial x}{\partial \xi}\vec{e}_x + \frac{\partial y}{\partial \xi}\vec{e}_y + \frac{\partial z}{\partial \xi}\vec{e}_z, \tag{2.27}$$

$$h_\eta \vec{e}_\xi = \frac{\partial x}{\partial \eta}\vec{e}_x + \frac{\partial y}{\partial \eta}\vec{e}_y + \frac{\partial z}{\partial \eta}\vec{e}_z \tag{2.28}$$

and

$$h_\phi \vec{e}_\xi = \frac{\partial x}{\partial \phi}\vec{e}_x + \frac{\partial y}{\partial \phi}\vec{e}_y + \frac{\partial z}{\partial \phi}\vec{e}_z \tag{2.29}$$

We notice that these vectors $\vec{e}_\xi$, $\vec{e}_\eta$, $\vec{e}_\phi$ are pairwise orthogonal in the sense that

$$\vec{e}_\eta \cdot \vec{e}_\xi = \vec{e}_\eta \cdot \vec{e}_\phi = \vec{e}_\xi \cdot \vec{e}_\phi = 0 \tag{2.30}$$

We can use these relationships to represent the vector $\vec{R}$ defined by equation (2.16) in terms of $\vec{e}_\xi$, $\vec{e}_\eta$, and $\vec{e}_\phi$. We see that

$$\vec{R} = (\vec{R}\cdot\vec{e}_\xi)\vec{e}_\xi + (\vec{R}\cdot\vec{e}_\eta)\vec{e}_\eta + (\vec{R}\cdot\vec{e}_\phi)\vec{e}_\phi \tag{2.31}$$

where

$$(\vec{R}\cdot\vec{e}_\xi) = \frac{x}{h_\xi}\cdot\frac{\partial x}{\partial \xi} + \frac{y}{h_\xi}\cdot\frac{\partial y}{\partial \xi} + \frac{z}{h_\xi}\cdot\frac{\partial z}{\partial \xi}, \tag{2.32}$$

$$(\vec{R}\cdot\vec{e}_\eta) = \frac{x}{h_\eta}\cdot\frac{\partial x}{\partial \eta} + \frac{y}{h_\eta}\cdot\frac{\partial y}{\partial \eta} + \frac{z}{h_\eta}\cdot\frac{\partial z}{\partial \eta}, \tag{2.33}$$

and

$$(\vec{R}\cdot\vec{e}_\phi) = \frac{x}{h_\phi}\cdot\frac{\partial x}{\partial \phi} + \frac{y}{h_\phi}\cdot\frac{\partial y}{\partial \phi} + \frac{z}{h_\phi}\cdot\frac{\partial z}{\partial \phi}, \tag{2.34}$$

First, substituting equations (2.18), (2.19), (2.20), (2.4), (2.5), and (2.6) into equation (2.32) we obtain

$$(\vec{R}\cdot\vec{e}_\xi) = \frac{d}{2}\cdot\xi\cdot\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}} \tag{2.35}$$

Next, determining that

$$\frac{\partial x}{\partial \eta} = \frac{d}{2}\left[(\xi^2+1)\right]^{1/2}(-\eta)\left[(1-\eta^2)\right]^{-1/2} cos(\phi) \tag{2.36}$$

and that

$$\frac{\partial y}{\partial \eta} = \frac{d}{2}\left[\sqrt{\frac{\xi^2+1}{1-\eta^2}}\right](-\eta)sin(\phi) \tag{2.37}$$

Equations (2.36), (2.37), and (2.4), (2.5), and (2.6) tell us that

$$(\vec{R}\cdot\vec{e}_\eta) = -\eta\frac{d}{2}\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}} \tag{2.38}$$

For a general coordinate transformation from an (x,y,z) frame to a (u,v,w) frame we have the relationship,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left\{\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right]\left(\frac{\partial u}{\partial t}\right)^2 + \right.$$

$$\left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]\left(\frac{\partial v}{\partial t}\right)^2 + \left[\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2\right]\left(\frac{\partial w}{\partial t}\right)^2$$

$$2\left[\frac{\partial x}{\partial u}\cdot\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}\right]\frac{du}{dt}\cdot\frac{dv}{dt} +$$

$$2\left[\frac{\partial x}{\partial u}\cdot\frac{\partial x}{\partial w} + \frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial w} + \frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial w}\right]\frac{du}{dt}\cdot\frac{dw}{dt} +$$

$$\left. 2\left[\frac{\partial x}{\partial v}\cdot\frac{\partial x}{\partial w} + \frac{\partial y}{\partial v}\cdot\frac{\partial y}{\partial w} + \frac{\partial z}{\partial v}\cdot\frac{\partial z}{\partial w}\right]\frac{dv}{dt}\cdot\frac{dw}{dt}\right\} \tag{2.39}$$

Making use of the orthogonality of the ξ, η, and ϕ coordinate system we see that with

$$(u,v,w) = (\xi,\eta,\phi) \tag{2.40}$$

that all of the terms in equation (2.39) with a factor of 2 vanish, and that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = h_\xi^2\left(\frac{d\xi}{dt}\right)^2 + h_\eta^2\left(\frac{d\eta}{dt}\right)^2 + h_\phi^2\left(\frac{d\phi}{dt}\right)^2 \tag{2.41}$$

Thus, for oblate spheroidal coordinates we obtain upon making use of equation (2.41) the following expressions for h_ξ, h_η, and h_ϕ. From this equation and equations (2.18), (2.19), and (2.20) we see that

$$h_\xi = \frac{d}{2}\sqrt{\frac{\xi^2+\eta^2}{\xi^2+1}} \tag{2.42}$$

Next observe that

$$h_\eta = \frac{d}{2}\sqrt{\frac{\xi^2+\eta^2}{1-\eta^2}} \tag{2.43}$$

Finally equations (2.23), (2.24), and (2.25) imply that

$$h_\phi = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2+1)} \tag{2.44}$$

In order to carry out vector calculus in oblate spheroidal coordinates we need the following relations. Equations (2.42), (2.43), and (2.44) imply that

$$h_\xi h_\eta h_\phi = \frac{d^3}{8}\left(\xi^2+\eta^2\right) \tag{2.45}$$

Also, equations (2.42), (2.43), and (2.44) imply that

$$\frac{h_\eta h_\phi}{h_\xi} = \frac{d}{2}\left(\xi^2+1\right) \tag{2.46}$$

The other two similar relations are

$$\frac{h_\xi \cdot h_\eta}{h_\phi} = \frac{d}{2}\left(\frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\right) \tag{2.47}$$

and

$$\frac{h_\xi \cdot h_\phi}{h_\eta} = \frac{d}{2}\left(1-\eta^2\right) \tag{2.48}$$

The above relations are needed to define the Helmholtz equation in oblate spheroidal coordinates. In order to define the *curl* operation in oblate spheroidal coordinates we need the product pairs as well Equations (2.42) and (2.43) imply that

$$h_\xi \cdot h_\eta = \frac{d^2}{4}\frac{\xi^2+\eta^2}{\sqrt{(\xi^2+1)(1-\eta^2)}} \tag{2.49}$$

Equations (2.42) and (2.44) imply that

$$h_\xi \cdot h_\phi = \frac{d^2}{4}\sqrt{(\xi^2+\eta^2)(1-\eta^2)} \tag{2.50}$$

Finally, equations (2.43) and (2.44) imply that

$$h_\eta \cdot h_\phi = \frac{d^2}{4}\sqrt{(\xi^2+\eta^2)(\xi^2+1)} \tag{2.51}$$

The curl operator in a general orthogonal coordinate system of orthogonal u, v, and w coordinates is given by

$$\begin{aligned} curl(\vec{E}) &= \\ &\frac{1}{h_v h_w}\left[\frac{\partial}{\partial v}(h_w E_w) - \frac{\partial}{\partial w}(h_v E_v)\right]\vec{e}_u + \\ &\frac{1}{h_u h_w}\left[\frac{\partial}{\partial w}(h_u E_u) - \frac{\partial}{\partial u}(h_w E_w)\right]\vec{e}_v + \\ &\frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}(h_v E_v) - \frac{\partial}{\partial v}(h_u E_u)\right]\vec{e}_w \end{aligned} \tag{2.52}$$

Equation (2.52) may be derived from combining the representation of Cartesian frame unit vectors in terms of $\vec{e}_u$, $\vec{e}_v$, and $\vec{e}_w$ and using the *gradient* equation,

$$grad(\Psi) = \frac{1}{h_u}\frac{\partial \Psi}{\partial u}\vec{e}_u + \frac{1}{h_v}\frac{\partial \Psi}{\partial v}\vec{e}_v + \frac{1}{h_w}\frac{\partial \Psi}{\partial w}\vec{e}_w \tag{2.53}$$

since (2.53) can used to express the curl of a vector field as the gradient cross this vector field. The divergence is given by

$$div(\vec{E}) = \left(\frac{1}{h_u h_v h_w}\right)\left\{\left(\frac{\partial}{\partial u}\right)(h_v h_w \cdot E_u)\right.$$

$$\left.\left(\frac{\partial}{\partial v}\right)(h_w h_u \cdot E_v) + \left(\frac{\partial}{\partial w}\right)(h_v h_u \cdot E_w)\right\} \tag{2.54}$$

It is easy to show that

$$curl(curl(\vec{E}) = grad(div(\vec{E}) - \Delta\vec{F} \tag{2.55}$$

where

$$\Delta\Psi = \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial\Psi}{\partial u}\right) + \right.$$

$$\left.\frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial\Psi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial\Psi}{\partial w}\right)\right\} \tag{2.56}$$

The relationship (2.55) implies that

$$\Delta(curl(\vec{E})) = - curl(curl(curl(\vec{E}))) = curl(\Delta(\vec{E})) \tag{2.57}$$

since

$$curl(grad(\Psi)) = \vec{0} \tag{2.58}$$

The Mie solution is based on applying the curl three times in succession to the vector $\vec{R}\Psi$ where

$$\vec{R} = grad\left(\frac{x^2+y^2+z^2}{2}\right) = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \tag{2.59}$$

which means that since for any vector field $\vec{F}$ and any scalar function Ψ it is true that

$$curl(\vec{F}\Psi) = \Psi curl(\vec{F}) + grad(\Psi)\times\vec{F} \tag{2.60}$$

that if $\vec{R}$ is defined by (2.59) that

$$curl(\Psi\vec{R}) = grad(\Psi)\times\vec{R} \tag{2.61}$$

and since We also have the relationship,

$$\Delta(\Psi\vec{R}) = \{\Delta(\Psi)\}\vec{R} + 2\cdot grad(\Psi) \tag{2.62}$$

Substituting equation (2.62) into equation (2.57) we see that in case Ψ satisfies

$$\Delta\Psi + k^2\Psi = 0 \tag{2.63}$$

that, since both the curl of a gradient and the divergence of a curl vanish,

$$curl(curl(curl(\Psi\vec{R}))) = curl(-\Delta\Psi) = + k^2 curl(\Psi\vec{R}) \tag{2.64}$$

Equation (2.64) is the basis of the Asano and Yamamoto solution ([1]) as well as the classical Mie solution for isotropic materials. For example, for an isotropic material we could let the electric vector be given by

$$\vec{E} = a \cdot \vec{M} + b \cdot \vec{N}, \tag{2.65}$$

where

$$\vec{M} = \frac{1}{k} \cdot curl(\vec{R}\Psi) \tag{2.66}$$

and where

$$\vec{N} = \frac{1}{k^2} \cdot curl(curl(\vec{R}\Psi)) \tag{2.67}$$

with Ψ being a solution of the scalar Helmholtz equation (2.63). Then if the magnetic vector $\vec{H}$ is defined by

$$\vec{H} = \frac{1}{-i\omega\mu}(k \cdot a \cdot \vec{N} + k \cdot b \cdot \vec{M}) \tag{2.68}$$

then the pair of vector valued functions (2.65) and (2.68) are solutions of both the Faraday and Ampere Maxwell equations for isotropic spheroids.

3 Vector Calculus for Oblate Spheroids

The Helmholtz operator in a general orthogonal ξ, η, and ϕ coordinate system may be expressed in the form is

$$\begin{aligned} \Delta\Psi + k^2\Psi &= \\ \frac{1}{h_\xi h_\eta h_\phi} \Bigg\{ \frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\phi}{h_\xi}\frac{\partial\Psi}{\partial\xi}\right) &+ \\ \frac{\partial}{\partial\eta}\left(\frac{h_\xi h_\phi}{h_\eta}\frac{\partial\Psi}{\partial\eta}\right) &+ \\ \frac{\partial}{\partial\phi}\left(\frac{h_\xi h_\eta}{h_\phi}\frac{\partial\Psi}{\partial\phi}\right)\Bigg\} &+ k^2\Psi \end{aligned} \tag{3.1}$$

and using the values of h_ξ, h_η, and h_ϕ for an oblate spheroidal coordinate system we have upon making the substitutions of equations (2.46), (2.48), (2.47), and (2.51) into equation (3.1) we deduce that

$$\begin{aligned} \Delta\Psi + k^2\Psi &= \\ \Bigg\{ \frac{\partial}{\partial\xi}\left((\xi^2+1)\frac{\partial\Psi}{\partial\xi}\right) &+ \frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial\Psi}{\partial\eta}\right) \\ &+ \frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2\Psi}{\partial\phi^2}\Bigg\} \\ &+ k^2\frac{d^2}{4}(\xi^2+\eta^2)\Psi = 0 \end{aligned} \tag{3.2}$$

We now seek solutions of equation (3.2) of the form

$$\Psi = R(\xi)S(\eta)exp(im\phi) \tag{3.3}$$

and substitute equation (3.3) into equation (3.2) and then divide all terms of this equation by the function Ψ defined by equation (3.3) after making use of the relationship

$$\frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)} =$$

$$\frac{1}{1-\eta^2} - \frac{1}{\xi^2+1} \tag{3.4}$$

and making the subsitution

$$c^2 = k^2d^2/4 \tag{3.5}$$

we obtain the relation,

$$\left\{\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial}{\partial\eta}S(c,\eta)\right)\right\}/S(c,\eta)$$

$$-\frac{m^2}{1-\eta^2} + c^2\eta^2 =$$

$$-\left\{\frac{\partial}{\partial\xi}\left((\xi^2+1)\frac{\partial R(c,\xi)}{\partial\xi}\right)\right\}/R(c,\xi) +$$

$$\frac{m^2}{\xi^2+1} + c^2\xi^2 = -\lambda_{(m,n)} \tag{3.6}$$

From equation (3.6) we obtain a kind of Rayleigh Ritz functional for the value of $\lambda_{(m,n)}$. Equation (3.6) tells us that

$$\lambda_{(m,n)} =$$

$$\left\{\int_{-1}^{1}\left[(1-\eta^2)\left(\frac{dS}{d\eta}\right)^2 + S^2\left\{(-c^2\eta^2)+\frac{m^2}{1-\eta^2}\right\}\right]d\eta\right\}/\left\{\int_{-1}^{1}S^2d\eta\right\} \tag{3.7}$$

We note that when c is equal to zero, we are dealing with a sphere and that the angular functions are the associated Legendre functions $P_n^m(\eta)$ so it makes sense that we want S to behave like the function $P_n^m(\eta)$ when c is zero. We note that either $n-m$ is even or odd, and we know the initial conditions exactly in each case. We use partial derivative notation for functions $G(c,\eta)$ and note that

$$D_2G(c,0) = \lim_{\eta\to 0}\frac{\partial G}{\partial\eta} \tag{3.8}$$

and define the intial conditions for the second order ordinary differential equation satisfied by the functions $S(c,\eta)$. We find that if $n-m$ is an even integer

$$S_{(m,n)}(c,0) = \left\{(-1)^{(n-m)/2}(n+m)!\right\}/\left\{2^n\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!\right\} \tag{3.9}$$

and

$$D_2S_{(m,n)}(c,0) = 0 \tag{3.10}$$

and when $n-m$ is an odd number that

$$S_{(m,n)}(c,0) = 0 \tag{3.11}$$

and that

$$D_2S_{(m,n)}(c,0) \;=\; \left\{(-1)^{(n-m-1)/2}(n+m+1)!\right\} / \left\{2^n\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!\right\} \tag{3.12}$$

With these initial conditions we have completely specified S and its partial derivative and mixed partial derivative as a function of η, c, and λ and we also know that

$$\lambda(0) \;=\; n(n+1) \tag{3.13}$$

This gives us an initial value problem and an ordinary differential equation

$$\lambda'(c) \;=\; F(c,\lambda) \tag{3.14}$$

where the function F is determined by differentiating both sides of equation (3.7) with respect to c and collecting terms involving $\lambda'(c)$, and then dividing all terms by the coefficient of $\lambda'(c)$ to get the first order ordinary differential equation (3.14). By the uniqueness of the Cauchy problem, different initial values cannot lead to the same eigenvalue at

$$c \;=\; k\cdot\frac{d}{2} \tag{3.15}$$

This is effective if c is real, but if k is complex, then we think of c as being a function of a paramter s defined by

$$c(s) \;=\; s\cdot k\cdot\frac{d}{2} \tag{3.16}$$

and with the same initial condition develop an ordinary differential equation of the form,

$$\lambda'(s) \;=\; G(s,\lambda) \tag{3.17}$$

Once these eigenfunctions are known, the steps for getting an exact solution for N layer isotropic spheroids is clear. The vector valued functions $\vec{M}$ described in the previous section are proportional to the curl of $\vec{R}\Psi$, where

$$\vec{R}\Psi \;=\; R_{\cdot}(\xi)S_{(m,n)}(\eta)exp(im\phi)\left\{\frac{d}{2}\cdot\left(\xi\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}\vec{e}_\xi \;-\; \eta\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}\vec{e}_\eta\right)\right\} \tag{3.18}$$

All three components appear in the $\vec{M}$ vector represented in spheroidal coordinates and this is given by

$$\begin{aligned} curl(\vec{R}\Psi) \;=\; & \vec{e}_\xi\left\{-im\eta\sqrt{\frac{1}{(\xi^2+1)(\xi^2+\eta^2)}}\Psi\right\} + \\ & \vec{e}_\eta\left\{im\xi\sqrt{\frac{1}{(1-\eta^2)(\xi^2+\eta^2)}}\Psi\right\} + \\ & \vec{e}_\phi\left\{\frac{\sqrt{(\xi^2+1)(1-\eta^2)}}{\xi^2+\eta^2}\left(\eta\frac{\partial}{\partial\xi} \;-\; \xi\frac{\partial}{\partial\eta}\right)\Psi\right\} \end{aligned} \tag{3.19}$$

This is $k\cdot\vec{M}$ and by expressing $\vec{N}$ in terms of the curl of $\vec{M}$, we obtain the vector fields, a combination of which can be used to represent the electric and magnetic vectors inside and outside the spheroid. The scattering problem is then solved by matching the η and the ϕ components of the electric and magnetic vectors across the boundaries of the spheroidal scatterer.

4 Prolate Spheroid Scattering – an Exact Solution

Here we consider a tensor material whose regions of continuity of tensorial electric permittivity and magnetic permeability are delimited by confocal spheroids. We assume that the foci are on the z-axis at $(0, 0, d/2)$ and $(0, 0, -d/2)$. We assume that the N confocal spheroids are defined by equations of the form,

$$\xi = \xi_i \tag{4.1}$$

where the relationship between Cartesian and Prolate spheroidal coordinates are given by equations

$$x = \frac{d}{2}\left[(1-\eta^2)(\xi^2-1)\right]^{1/2} cos(\phi) \tag{4.2}$$

and

$$y = \frac{d}{2}\left[(1-\eta^2)(\xi^2-1)\right]^{1/2} sin(\phi) \tag{4.3}$$

and by equation (2.6) which is the same in oblate and prolate coordinates, which means that the equation of the ith spheroid (4.1) is, in Cartesian coordinates given by

$$\frac{x^2+y^2}{(d^2/4)(\xi^2-1)} + \frac{z^2}{(d\cdot\xi)^2/4} \;=\; = 1 \tag{4.4}$$

We use the curl operator in a coordinate system with the same angle coordinate ϕ of spherical coordinates that runs from 0 to 360 degrees so that in spherical, spheroidal, cylindrical, or toroidal coordinates the Faraday Maxwell equation is defined by

$$\begin{aligned} curl(\vec{E}) \;=\; & \frac{1}{h_\eta h_\phi}\left[\frac{\partial}{\partial\eta}(h_\phi E_\phi) \;-\; imh_\eta E_\eta\right]\vec{e}_\xi + \\ & \frac{1}{h_\xi h_\phi}\left[imh_\xi E_\xi \;-\; \frac{\partial}{\partial\xi}(h_\phi E_\phi)\right]\vec{e}_\eta + \\ & \frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial\xi}(h_\eta E_\eta) \;-\; \frac{\partial}{\partial\eta}(h_\xi E_\xi)\right]\vec{e}_\phi \;= \\ & -\; i\omega\mu_\xi H_\xi\vec{e}_\xi \;-\; i\omega\mu_\eta H_\eta\vec{e}_\eta \;-\; i\omega\mu_\phi H_\phi\vec{e}_\phi \end{aligned} \tag{4.5}$$

We can solve equation (4.5) for components of the magnetic vector; this is simply a statement of Faraday's law which says that if one integrates the tangential component of the electric vector around the boundary of a surface, the value is equal to the negative time derivative of the normal component of the magnetic flux $\vec{B}\cdot\vec{n}$ integrated over the surface.

Ampere's law states that if we integrate the tangential component of the magnetic vector around the boundary of a surface that this is equal to the normal component of the current, which includes displacement current or the time derivative of the vector $\vec{D}$) as well as conduction current $\vec{J}$, integrated over the surface. The Ampere Maxwell equation is, therefore, in this coordinate system, given by

$$curl(\vec{H}) \;=\; \frac{1}{h_\eta h_\phi}\left[\frac{\partial}{\partial\eta}(h_\phi H_\phi) \;-\; imh_\eta H_\eta\right]\vec{e}_\xi +$$

$$\frac{1}{h_\xi h_\phi}\left[imh_\xi H_\xi \; - \; \frac{\partial}{\partial \xi}(h_\phi H_\phi)\right]\vec{e}_\eta \; +$$

$$\frac{1}{h_\xi h_\eta}\left[\frac{\partial}{\partial \xi}(h_\eta H_\eta) \; - \; \frac{\partial}{\partial \eta}(h_\xi H_\xi)\right]\vec{e}_\phi \; =$$

$$(i\omega\epsilon_\xi + \sigma_\xi)E_\xi\vec{e}_\xi \; + \; (i\omega\epsilon_\eta + \sigma_\eta)E_\eta\vec{e}_\eta \; + \; (i\omega\epsilon_\phi + \sigma_\phi)E_\phi\vec{e}_\phi \tag{4.6}$$

Solving equation (4.5) for $\vec{H}_\xi$ we see that

$$\vec{H}_\xi \; = \; \frac{i}{\omega\mu_\xi}\left(\frac{1}{h_\eta h_\phi}\right)\left[\frac{\partial}{\partial \eta}(h_\phi E_\phi) \; - \; imh_\eta E_\eta\right] \tag{4.7}$$

Equating the η components of both sides of the Ampere Maxwell equation (4.6) and substituting equation (4.7) into this equation we deduce that

$$\left[(i\omega\epsilon_\eta + \sigma_\eta) + \frac{1}{h_\xi h_\phi}\left[imh_\xi\left\{\frac{i}{\omega\mu_\xi}\left(\frac{1}{h_\eta h_\phi}\right)imh_\eta\right\}\right]\right]E_\eta \; =$$

$$\frac{1}{h_\xi h_\phi}\left[imh_\xi\left\{\frac{i}{\omega\mu_\xi}\left(\frac{1}{h_\eta h_\phi}\right)\frac{\partial}{\partial \eta}(h_\phi E_\phi)\right\} \; - \; \frac{\partial}{\partial \xi}(h_\phi H_\phi)\right] \tag{4.8}$$

We can introduce functions A_η and B_η such that equation (4.8) may be rewritten as

$$E_\eta \; = \; A_\eta\frac{\partial}{\partial \eta}(h_\phi E_\phi) \; - \; B_\eta\frac{\partial}{\partial \xi}(h_\phi H_\phi) \tag{4.9}$$

Similarly, it is clear that by equating the η components of both sides of (4.5) and solving the Ampere Maxwell equation (4.6) for E_ξ we see that we can find functions F_η and G_η such that

$$H_\eta \; = \; F_\eta\frac{\partial}{\partial \eta}(h_\phi H_\phi) \; - \; G_\eta\frac{\partial}{\partial \xi}(h_\phi E_\phi) \tag{4.10}$$

where if we define $k(\mu_\eta, \epsilon_\xi)$ by the rule,

$$k(\mu_\eta, \epsilon_\xi)^2 \; = \; (\omega^2\mu_\eta\epsilon_\xi - i\omega\mu_\eta\sigma_\xi) \tag{4.11}$$

then

$$F_\eta \; = \; \frac{im/h_\eta}{m^2 + k(\mu_\eta, \epsilon_\xi)^2 \cdot h_\phi^2} \tag{4.12}$$

We now solve equation (4.6) for E_ξ obtaining the relationship

$$E_\xi \; = \; \frac{1}{i\omega\epsilon_\xi + \sigma_\xi}\left[\frac{1}{h_\eta h_\phi}\right]\left[\frac{\partial}{\partial \eta}(h_\phi H_\phi) \; - \; imh_\eta H_\eta\right]$$

We can, thus, express E_ξ in terms of E_ϕ and H_ϕ and can similarly express H_ξ. If we make all of these substitutions into the $\vec{e}_\phi$ components of both sides of the Faraday and Ampere Maxwell equations we get two coupled partial differential equations in the E_ϕ and H_ϕ variables.

Thus, we have a coupled system of elliptic equations of second order in the angular components of the electric and magnetic vectors, with all other components of the electric and magnetic vectors being simply expressed in terms of these components.

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RAPID MATRIX INVERSION

D. K. Cohoon

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Contents

1 INTRODUCTION

We consider the problem of solving linear equations where the coefficients and unknowns may be members of $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. The dramatic difference between an $N^{2+\epsilon}$ matrix inversion method and the traditional method which requires N^3 steps to invert an N by N matrix can be seen by conceiving of a computer that could perform ten billion operations per second. Suppose one wanted to solve an integral equation describing the interaction of a complex scatterer with a complex electromagnetic radiation field. One usually divides the scatterer into small subunits within which one can approximate the induced fields with a low degree polynomial. If one had a model of a man or some other scatterer divided into a million small subunits in which one approximates electric and magnetic fields components by polynomials of degree two, then roughly 1000 human lifetimes would be required to solve for these unknowns once the matrix was created using the traditional method, whereas only one hour would be needed for the method requiring $N^{2+\epsilon}$ steps, if ϵ is sufficiently small.

We describe an order $N^{2+\epsilon}$ algorithm for inverting a class of dense matrices. Part of our algorithm is based on rapid multiplication. The first algorithm which gave an order smaller than N^3 for multiplying matrices was due to Strassen ([6]). This algorithm was based on reducing the problem using 2 by 2 submatrices and the multiplication scheme

$$\begin{pmatrix} a_{(1,1)} & a_{(1,2)} \\ a_{(2,1)} & a_{(2,2)} \end{pmatrix} \begin{pmatrix} b_{(1,1)} & b_{(1,2)} \\ b_{(2,1)} & b_{(2,2)} \end{pmatrix} =$$

$$\begin{pmatrix} c_{(1,1)} & c_{(1,2)} \\ c_{(2,1)} & c_{(2,2)} \end{pmatrix} \tag{1.1}$$

where

$$c_{(1,1)} = p_1 + p_6 - p_5 + p_7 \tag{1.2}$$

$$c_{(1,2)} = p_3 + p_5 \tag{1.3}$$

$$c_{(2,1)} = p_2 + p_6 \tag{1.4}$$

$$c_{(2,2)} = p_1 - p_2 + p_3 + p_4 \tag{1.5}$$

and the functions p_j for

$$j \in \{1, 2, 3, 4, 5, 6, 7\}$$

are defined by the equations

$$p_1 = (a_{(1,1)} + a_{(2,2)})(b_{(1,1)} + b_{(2,2)}) \tag{1.6}$$

and

$$p_2 = (a_{(2,1)} + a_{(2,2)})b_{(1,1)} \tag{1.7}$$

$$p_3 = a_{(1,1)}(b_{(1,2)} - b_{(2,2)}) \tag{1.8}$$

$$p_4 = (-a_{(1,1)} + a_{(2,1)})(b_{(1,1)} + b_{(1,2)}) \tag{1.9}$$

$$p_5 = (a_{(1,1)} + a_{(1,2)})b_{(2,2)} \tag{1.10}$$

$$p_6 = a_{(2,2)}(-b_{(1,1)} + b_{(2,1)}) \tag{1.11}$$

$$p_7 = (a_{(1,2)} - a_{(2,2)})(b_{(2,1)} + b_{(2,2)}) \tag{1.12}$$

For example, in checking equation (1.2) observe that

$$\begin{aligned} p_1 + p_6 - p_5 + p_7 &= (a_{(1,1)} + a_{(2,2)})(b_{(1,1)} + b_{(2,2)}) + \\ a_{(2,2)}(-b_{(1,1)} + b_{(2,1)}) &- (a_{(1,1)} + a_{(1,2)})b_{(2,2)} + \\ (a_{(1,2)} - a_{(2,2)})(b_{(2,1)} + b_{(2,2)}) &= a_{(1,1)}b_{(1,1)} + a_{(1,2)}b_{(2,1)} \end{aligned} \tag{1.13}$$

Here there are 7 multiplications and 18 additions and subtractions. We consider a 2^N by 2^N matrix. For this size of matrix we suppose that $\mathcal{A}(N)$ is the number of additions and that $\mathcal{M}(N)$ is the number of multiplications. Then the number of multiplications is

$$\mathcal{M}(N+1) = 7\mathcal{M}(N) \tag{1.14}$$

and the number of additions is

$$\mathcal{A}(N+1) = 18(2^N)(2^N) + 7\mathcal{A}(N) \tag{1.15}$$

since there are

$$4^N = (2^N)(2^N) \tag{1.16}$$

entries in a 2^N by 2^N matrix. We now make use of the following lemma.

Lemma 1.1 *The solution of the difference equation*

$$y_{n+1} = A[R^n] + \lambda y_n$$

that satisfies $y_0 = 0$ is when R is distinct from λ given by

$$y_n = \left(\frac{A}{\lambda - R}\right)(\lambda^n - R^n)$$

Solving the difference equation we see that since a general solution is

$$\mathcal{A}(N) = C(7^N) - 6(4^N) \tag{1.17}$$

since a particular solution is, upon using $A = 18$ and $\lambda = 7$ and $R = 4$ in the Lemma, given by

$$\mathcal{P}(N) = -6(4^N) \tag{1.18}$$

As a check note that

$$-6(4)(4^N) = 18(4^N) + 7(-6)(4^N). \tag{1.19}$$

Since

$$\mathcal{A}(0) = 0 \tag{1.20}$$

as no additions are required for multiplying 1 by 1 matrices, we see that

$$C = 6 \tag{1.21}$$

We can then see that the total number of operations for multiplying a 2^N by 2^N matrix is estimated by the inequality

$$\mathcal{A}(N) + \mathcal{M}(N) < 6 \cdot 7^N \ + \ 7^N \tag{1.22}$$

Thus, since the number of rows is

$$M = 2^N \tag{1.23}$$

we see an exponent α such that

$$M^\alpha = 2^{\alpha N} < 7^{N+1} \tag{1.24}$$

or applying log_2 to both sides of inequality (1.24) we see that the order α is estimated by the inequality,

$$\alpha N < log_2(7) \ + \ N\left\{log_2(7)\right\} \tag{1.25}$$

or upon dividing all terms of equation (1.25) by N and letting N go to infinity, we see that asymptotically the order α is estimated by

$$\alpha < log_2(7) = 2.807 \tag{1.26}$$

which means that the number of operations for multiplying two M by M matrices is about $M^{2.807}$.

The number of steps required for matrix inversion can in many circumstances be shown to be quivalent, except for multiplication by a constant independent of the number of rows, to the number of operations required for matrix multiplication.

Proposition 1.1 *If the nonsingular N by N matrix A can be expressed in the form $A = D - P$, where D is an invertible diagonal matrix, and if $D^{-1}P$ has ℓ_∞ matrix norm less than 1, then for every number n of significant digits, there is a number M_n, independent of the number N of rows in the matrix, such that the entries of the inverse A^{-1} of A can be determined to at least n significant digits by the sum,*

$$A_n^{-1} = D^{-1}\left[I + (D^{-1}P) + (D^{-1}P)^2 + \cdots (D^{-1}P)^{M_n}\right] \tag{1.27}$$

which means that the number of operations for computing the inverse is equal to

$$C(A^{-1}) = (M_n - 1)C_N + \{M_n + 1\}\, N^2 \tag{1.28}$$

where C_N denotes the number of operations required for matrix multiplication.

Proof. Observe that since if r is equal to the norm of $D^{-1}P$ that the norm of the remainder can be made arbitrarily small simply by making M_n large enough, since the remainder terms involve sums of powers of $D^{-1}P$ and

$$\| (D^{-1}P)^{M_n} \| \leq r^{M_n} < 10^{-n} \| D^{-1} \| \tag{1.29}$$

that then A_n^{-1} agrees with A^{-1} to n decimal places. Since D is diagonal, N^2 operations are required for the computation of $D^{-1}P$. Let us suppose that C_N operations are required for matrix multiplication. Then by performing the calculations in succession we see that exactly C_N operations are required for each of the computations $(D^{-1}P)^2, (D^{-1}P)^3, \cdots$, and $(D^{-1}P)_n^M$ giving a total of $(M_n - 1)C_N$ operations. There are $M_n N^2$ summation operations followed by N^2 operations for the multiplication of the sum by D^{-1}. This argument proves equation (1.28).

Proposition 1.2 *If A is any endomorphism of R^n, then there is a*

$$\lambda_0 \in \mathbf{C} - \{0\}$$

such that

$$T_0 = \lambda_0 I - A \tag{1.30}$$

is invertible and

$$R_0 = (-T_0)^{-1} \tag{1.31}$$

is known. Then there is an open connected set Ω containing λ_0 such that

$$R_\lambda = (\lambda I - T_0)^{-1} \tag{1.32}$$

exists and

$$\frac{dR_\lambda}{d\lambda} = -R_\lambda^2 \tag{1.33}$$

with

$$R_0 = (-T_0)^{-1} \tag{1.34}$$

giving the initial condition for the Riccati equation (1.33).

Proof. We use a classical theory of resolvents of operators (e.g. Friedman [3], pp 194-195). To find a λ_0 such that the ℓ_∞ norm of the matrix $\lambda_0^{-1}A$ is smaller than 1 and use the result of the previous proposition.

Definition 1.1 *We say that a* $\lambda \in \mathbf{C}$ *is in the resolvent set of* T_0 *if*

$$R_\lambda = (\lambda I - T_0)^{-1} \tag{1.35}$$

exists.

Note that we can consider the component of the complex plane $\mathbf{C}$ containing λ_0 such that the matrix R_λ defined by equation (1.35) exists. We know that T_0 is invertible, and that if λ is in the resolvent set of T_0, then

$$Det(\lambda I - T_0) \neq 0 \tag{1.36}$$

and that the set of all λ satisfying equation (1.36) is, since the determinant function Det is continuous, an open set U containing λ_0. We simply choose Ω to be a convex open subset of the component of U containing λ_0. We then write

$$\begin{aligned} R_\lambda \; - \; R_\mu &= (\lambda I - T_0)^{-1} \; - \; (\mu I - T_0)^{-1} \\ (\lambda I - T_0)^{-1}&[(\mu I - T_0) - (\lambda I - T_0)](\mu I - T_0)^{-1} \\ (\mu - \lambda)&(\lambda I - T_0)^{-1}(\mu I - T_0)^{-1} \\ &= (\mu \; - \; \lambda) R_\lambda R_\mu \end{aligned} \tag{1.37}$$

Dividing both sides of equation (1.37) by $\lambda - \mu$ and taking the limit as μ approaches λ we see that equation (1.33) is valid in Ω since for all choices of λ and μ in Ω the straight line joining λ and μ is also contained in Ω. This completes the proof of the proposition.

A contour integration method can also be used to calculate the inverse of A. One method is described by the following theorem.

Theorem 1.1 *If C is a simple closed curve in the complex plane which contains λ_0 but does not cross or contain within its interior any of the points λ where R_λ does not exist, then*

$$A^{-1} = \frac{1}{2\pi i} \int_C \frac{R_\lambda}{\lambda - \lambda_0} d\lambda \tag{1.38}$$

A variety of identities similar to that of equation (1.37) are found in Cohoon ([1]). We note that ([1]) if we introduce the operator,

$$R_a f(x) = \int_0^x exp(a(x - s))f(s)ds, \tag{1.39}$$

we can systematically solve ordinary differential equations of the form,

$$\left[\left(\frac{d}{dx} - \lambda_1\right)^{n_1} \left(\frac{d}{dx} - \lambda_2\right)^{n_2} \cdots \left(\frac{d}{dx} - \lambda_r\right)^{n_r}\right] u(x) \; = \; f(x), \tag{1.40}$$

where $f(x)$ is any member of $C^\infty(\mathbb{R})$. A solution of the general nonhomogeneous linear ordinary differential equation (1.40) with constant coefficients is given by

$$u(x) \;=\; \left[R^{n_1}_{\lambda_1} R^{n_2}_{\lambda_2} \cdots R^{n_r}_{\lambda_r}\right] f(x) \tag{1.41}$$

where the latter iterated integral expression can be reduced to a sum of single integrals by realizing that the identity

$$R_\lambda R_\mu = \frac{R_\lambda - R_\mu}{\lambda - \mu} \tag{1.42}$$

is identical to the expansion by partial fractions relation

$$X_\lambda X_\mu = \frac{X_\lambda - X_\mu}{\lambda - \mu} \tag{1.43}$$

where X_λ and X_μ are defined by

$$X_\lambda = \frac{1}{x - \lambda} \tag{1.44}$$

Iterative methods have been suggested by many authors for overcoming computational complexity in electromagnetic interaction computations and most of these developments required that the scattering body be diaphanous or have nearly the properties of free space. However, a way of obtaining discretizations that use lower order matrices and use manipulations of these matrices to improve the accuracy of the solution have been developed by Cohoon ([2]); this paper describes a computerizable finite rank integral equation whose solution is exactly the projection onto the space of approximates of the solution of the original infinite rank integral equation. These methods of obtaining discretizations of integral equations that, upon solution, give accurate solutions of scattering problems, coupled with the potentially rapid methods of multiplying and inverting matrices give some promise of being able to develop realistic electromagnetic radiation dosimetry models as well as solving much larger matrix modeling problems of all types.

2 THE MATRIX INVERSION ALGORITHM

We now produce a method of finding the inverse of a general matrix A using these ideas.

Theorem 2.1 *If A is an automorphism of $\mathbb{C}^N$ or $\mathbb{R}^N$ and λ is a complex number such that*

$$R_\lambda \;=\; (\lambda I - T_0)^{-1}$$

exists, then we can find a connected, convex open set cotaining both λ_0 and $\lambda = 0$ such that if the R_λ defined by equation (2.1) exists, then we can obtain A^{-1} by solving the ordinary differential equation

$$F'(s) \;=\; \frac{dR_\lambda}{d\lambda}\frac{d\lambda}{ds} \tag{2.1}$$

along a curve

$$\lambda : [0,1] \to \mathbb{C} \tag{2.2}$$

joining $\lambda = 0$ to $\lambda = \lambda_0$ where

$$F(0) \;=\; R_0 \;=\; (-T_0)^{-1} \tag{2.3}$$

is the initial condition for equation (2.1).

Proof. We just have to pick a curve $\lambda = \lambda(s)$ which does not pass through any points of the spectrum, where

$$Det(\lambda I - T_0) \;=\; 0 \tag{2.4}$$

and since there are only a finite number of these points, we could with the proper choice of λ_0 find a straight line

$$\lambda(s) \;=\; s\lambda_0 \tag{2.5}$$

which has this property. Then the differential equation (2.1) has the form,

$$F'(s) \;=\; \frac{dR_\lambda}{d\lambda}\frac{d\lambda}{ds} \;=\; -R_\lambda^2\lambda_0 \tag{2.6}$$

This completes the proof of the theorem.

A variant of the ideas of this theorem can be used to treat a class of infinite matrices arising in several areas, including electromagnetic scattering theory.

Theorem 2.2 *If L is a Fredholm integral operator given by*

$$LE(q) = \int_\Omega \mathcal{G}(p,q)E(q)d\nu(q) \tag{2.7}$$

associated with the Fredholm integral equation

$$E(p) - E^i(p) = \lambda(LE)(p) \tag{2.8}$$

where E^i is known and we are seeking the function E, and $\mathcal{R}_\lambda$ is the resolvent kernel defined by

$$\frac{d\mathcal{R}}{d\lambda} \;=\; \int_\Omega \mathcal{R}_\lambda(p,w)\mathcal{R}_\lambda(w,q)d\nu(w) \tag{2.9}$$

with the initial condition being,

$$\mathcal{R}_0 \;=\; \mathcal{G}(p,q) \tag{2.10}$$

then the solution of equation (2.8) is given by

$$E = E^i + \lambda R_\lambda E^i \tag{2.11}$$

where

$$R_\lambda(p)E^i(p) = \int_\Omega \mathcal{R}_\lambda(p,q)E^i(q)d\nu(q) \tag{2.12}$$

Proof: The differential equation (2.9) is derived by expressing the inverse integral operator as a Born series assuming that λ is small and then analytically continuing the solution.

3 RAPID MATRIX INVERSION

In the previous sections we have related matrix inversion speed to matrix multiplication speed for a class of dense matrices. We shall use the lines for extending the matrix multiplication speed improvement for 2 by 2 matrices in our introductory section.

Assume that we have developed a matrix multiplication method for two m by m matrices that uses p multiplications and q additions.

Theorem 3.1 *(Fast Matrix Multiplication) Suppose that A is a matrix with $M = m^N$ rows and the same number of columns. Suppose that $\mathcal{A}(N)$ is the number of additions and $\mathcal{M}(N)$ is the number of multiplications required for multiplying two matrices of this size. Then*

$$\mathcal{M}(N+1) = p\mathcal{M}(N) \tag{3.1}$$

and

$$\mathcal{A}(N+1) = q(m^N)(m^N) + p\mathcal{A}(N) \tag{3.2}$$

and if α is an exponent such that the matrix multiplication requires M^α steps for multiplying two M by M matrices, then

$$\alpha \; < \; Log_m(p) \tag{3.3}$$

which means to get the desired result the number of multiplications required for the matrix multiplication just has to get down to

$$p \; = \; m^{2+\epsilon} \tag{3.4}$$

proof: To derive the difference equations for the number of multiplications and additions, we simply subdivide two m^{N+1} by m^{N+1} matrices into m submatrices each of which are m^N by m^N matrices. Treating the submatrices as members of an algebra, we see that the number of additions required to carry out the multiplications is q times the the number of entries, m^{2N} plus the number of multiplications p of of submatrices times the number $\mathcal{A}(N)$ of additions used in each of these multiplications. The number of multiplications is simply p times the number $\mathcal{M}(N)$ of multiplications required for multiplying two of the submatrices. These arguments constitute a derivation of the difference equations (3.1) and (3.2).

We now give the solution of the difference equations (3.1) and (3.2) and an estimate of $\mathcal{A}(N) + \mathcal{M}(N)$. Thinking of particular solutions plus general solutions of the homogenous equation associated with equation (3.2), we see that one solution of equation (3.2) is Dm^{2N} and that substitution into equation (3.2) implies that

$$Dm^{2(N+1)} - pDm^{2N} = qm^{2N} \tag{3.5}$$

which implies that

$$m^{2N}\left\{Dm^2 - pD - q\right\} \; = \; 0, \tag{3.6}$$

which tells us that

$$D = \frac{q}{m^2 - p} \tag{3.7}$$

Thus, the most general solution of equation (3.2) is

$$\mathcal{A}(N) = Cp^N + \left(\frac{q}{m^2 - p}\right) m^{2N} \tag{3.8}$$

We can solve for the constant C by considering a matrix with one row and column, the case $N = 0$, which would since no additions are required, tell us that

$$0 \;=\; \mathcal{A}(N) \;=\; C + \left(\frac{q}{m^2 - p}\right) \tag{3.9}$$

or that

$$C = \frac{q}{p - m^2} \tag{3.10}$$

As a check on this work, we also consider the situation where the number of rows is $M = m^1$ which means that $N = 1$. We know that in this case

$$q \;=\; \mathcal{A}(1) = Cp + \left(\frac{q}{m^2 - p}\right) m^2 \tag{3.11}$$

which implies that

$$C = \frac{1}{p}\left[q + \left(\frac{q}{p - m^2}\right) m^2\right] \tag{3.12}$$

which after some manipulation is seen to be identical to that given by equation (3.10). The number of multiplications is easily seen to be

$$\mathcal{M}(N) = p^N \tag{3.13}$$

We now estimate the total number of operations required to carry out the matrix multiplication. Observe that

$$\mathcal{A}(N) + \mathcal{M}(N) = \left(\frac{q}{p - m^2}\right) [p^N - m^{2N}] + p^N \tag{3.14}$$

From equation (3.14) we see that

$$\mathcal{A}(N) + \mathcal{M}(N) < \left[\frac{q}{p - m^2} \;+\; 1\right] p^N \tag{3.15}$$

Observe that if $M = m^N$ is the number of rows and if the number of operations for matrix multiplication is M^α, then equation (3.15) implies that

$$M^\alpha \;=\; m^{\alpha N} \;<\; \left[\frac{q}{p - m^2} \;+\; 1\right] p^N \tag{3.16}$$

Taking the logarithm to the base m of both sides of equation (3.16) we deduce that

$$\alpha N Log_m(m) \;<\; Log_m\left[1 + \frac{q}{p - m^2}\right] + N Log_m(p) \tag{3.17}$$

Dividing all terms of equation (3.17) by N and taking the limit as N approaches infinity we see that asymptotically we need

$$\alpha < Log_m(p) \tag{3.18}$$

If we could carry out the operations with $p = m^{2+\epsilon}$, then we would have

$$M^{\alpha} = M^{2+\epsilon} \tag{3.19}$$

as an estimate of the number of operations required to multiply two M by M matrices. The number of additions used in the multiplication of the submatrices apparently makes no difference the the asymptotic computational complexity of the matrix multiplication of the large matrices. Our discussion in the previous section then gives us a class of dense matrices that can also be inverted in this number of operations.

4 Fields of Positive Characteristic

For fields of positive characteristic which do not have a transcendental element the following theorem shows that matrix inverses can be systematically obtained by repeated multiplication, and picking the first two distinct matrix powers which are identical. We characterize the fields for which this is possible.

Theorem 4.1 *If a field F has characteristic zero, then there is an invertible square matrix $\overline{\overline{M}}$ having the property that for every postive integer r and every nonnegative integer s*

$$\overline{\overline{M}}^{r+s} \neq \overline{\overline{M}}^{s} \tag{4.1}$$

If the field F has positive characteristic equal to a prime p and contains one transcendental element x, then we can also find an invertible square matrix $\overline{\overline{M}}$ filled with elements of F such that for every positive integer r and every nonnegative integer s inequality (4.1) is valid. For the remaining case where the (possibly infinite) field F has characteristic p and every element of F is algebraic over the ground field F_p containing p elements, then if a square matrix $\overline{\overline{M}}$ over F contains a finite number of rows, then we can find a positive integer r and a nonnegative integer s such that

$$\overline{\overline{M}}^{r+s} = \overline{\overline{M}}^{s} \tag{4.2}$$

Proof of the theorem. To see the validity of inequality (4.1) for the characterisitic zero case we simply consider the invertible one by one matrix,

$$\overline{\overline{M}} = (2) \tag{4.3}$$

and observe that distinct powers of 2 are not equal. The validity of inequality (4.1) when the field contains an element x which is transcendental over the ground field F_p containing a prime number p elements is seen by considering the one by one matrix

$$\overline{\overline{M}} = (x) \tag{4.4}$$

and observing that if it were true that for some positive integer r and some nonnegative integer s we had

$$x^{r+s} = x^s \tag{4.5}$$

then x would be algebraic and not transcendental, a contradiction.

To take care of the final case where all elements of the field are algebraic over the p element ground field $\mathbb{F}_p$ and $\overline{\overline{M}}$ is a square matrix with m rows, we let $\mathbb{K}$ be the smallest algebraic extension of the finite $\mathbb{F}_p$ which contains all elements of the matrix $\overline{\overline{M}}$. We let n be the number of elements of the finite field $\mathbb{K}$. Then

$$cardinality\left\{\overline{\overline{M}}\ : M_{(i,j)}\in\mathbb{K},\ i,j\in\{1,2,\cdots,m\}\right\} = n^{m^2} \tag{4.6}$$

Hence, as all powers of the matrix $\overline{\overline{M}}$ belong to a finite set of matrices, two of these powers, say the sth power and the tth power where $t > s$ must be the same which means that

$$\overline{\overline{M}}^t = \overline{\overline{M}}^s \tag{4.7}$$

We then let $r = t - s$. obtaining equation (4.2).

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An Exact Solution of Mie Type for Scattering by a Multilayer Anisotropic Sphere

D. K. Cohoon

Department of Mathematics 033-16
Temple University
Philadelphia, PA 19122, USA

Abstract– The purpose of this paper is to describe methods of resolving discrepancies between experimental observations of scattering by crystalline particles and attempts to explain these observations assuming that the electrical properties of these particles can be described by the use of scalar valued functions for the permittivity, conductivity, and permeability. We can develop coupled integral equations describing the interaction of electromagnetic radiation with a heterogeneous, penetrable, dispersive, anisotropic scatterer and can use several methods of solving these integral equations. The solution of the problem of describing scattering by an anisotropic sphere can be substituted into the integral equation to check the integral equation formulation of the problem. Conditions are given for the uniqueness of the solution of the associated transmission problem. Because of the multiple propagation constants in an anisotropic material, the trivial uniqueness arguments valid for isotropic scatterers do not have a guaranteed success in understanding the more complex interaction phenomena.

An exact analytical Mie-like solution has been obtained for fields induced in and scattered by an N layer sphere, where each layer has anisotropic constitutive relations. We show that as the tensor parameters change so that each layer becomes isotropic, then the distinct radial functions used in representing the electric and magnetic fields induced in the structure both converge to the same spherical Bessel and Hankel functions and all the propagation constants in each layer converge to the propagation constant k given by

$$k^2 = \omega^2 \mu \epsilon - i\omega\mu\sigma$$

and the solution approaches the ordinary Mie solution for an N-layer sphere. The anisotropic sphere computer code, for the case of magnetic losses, dielectric losses, and dissipative impedance sheets, and perfectly conducting or penetrable inner cores has been validated by energy balance computations involving balancing the difference between the total energy entering the sphere minus the total energy scattered away with the sum of the surface integrals representing losses due to dissipative impedance sheets separating the layers plus the sum of the triple integrals over the layers whose values represent magnetic and dielectric losses within the anisotropic penetrable layers.

Two Bessel functions with two different complex indices depending on ratios of tangential and radial magnetic properties and ratios of tangential and radial electrical properties, respectively, participate in the solution in the case of scattering by the simplest anisotropic sphere. The scattering problem is solved for the case where the scatterer consists of (i) N anisotropic dielectric layers, (ii) N of these layers separated by sheets of charge or impedances, and (iii) a perfectly conducting core surrounded by $N-1$ anisotropic layers.

1. INTRODUCTION

The atmosphere of the earth is filled with small anisotropic scatterers. It is im-

portant to understand the nature of the scattering of light for individual aerosol particles, to determine precisely how a cloud including these particles might impede the progress of the sunlight or laser communication. These anisotropic scatterers have many sources including volcanic eruptions and human activity. The term anisotropic here refers to the constitutive relations between $\overline{B}$ and $\overline{H}$ and between $\overline{D}$ and $\overline{E}$, and $\overline{J}$ and $\overline{E}$ which are tensorial; in isotropic particles the Fourier transforms with respect to time of these quantities are related by scalars. We will use $\overline{\overline{\epsilon}}$ to denote the tensor permittivity of the anisotropic particle, and we let ϵ_0 denote the permittivity of free space. We let $\overline{\overline{I}}$ denote the 3 by 3 identity matrix. The tensor magnetic permeability is denoted by $\overline{\overline{\mu}}$. We consider time harmonic radiation with frequency ω. By lumping together the frequency times the imaginary part of the permittivity tensor and the real part of the usual conductivity tensor we get a real tensor $\overline{\overline{\sigma}}$. The Maxwell equations for anisotropic materials, therefore have the form

$$\nabla \times \overline{H} = i\omega\epsilon_0\overline{E} + \left[i\omega(\overline{\overline{\epsilon}} - \epsilon_0\overline{\overline{I}}) + \overline{\overline{\sigma}}\right] \cdot \overline{E} \tag{1.1}$$

and

$$\nabla \times \overline{E} = -i\omega\mu_0\overline{H} - i\omega(\overline{\overline{\mu}} - \mu_0\overline{\overline{I}}) \cdot \overline{H} \tag{1.2}$$

Equations (1.1) and (1.2) together with the fact that the divergence of a curl is zero tell us that

$$\frac{\nabla \cdot \left[(\overline{\overline{\epsilon}} - \epsilon_0\overline{\overline{I}} - i\overline{\overline{\sigma}}/\omega) \cdot \overline{E}\right]}{\epsilon_0} = -\nabla \cdot \overline{E} \tag{1.3}$$

Similarly, the fact that

$$\nabla \cdot (\overline{\overline{\mu}} \cdot \overline{H}) = 0 \tag{1.4}$$

implies that

$$\nabla \cdot (\overline{\overline{\mu}} \cdot \overline{H} - \mu_0\overline{H}) = -\mu_0\nabla \cdot \overline{H} \tag{1.5}$$

Thus, thinking in terms of the traditional free space Maxwell equations with electric and magnetic currents $\overline{J}_e$ and $\overline{J}_m$ and electrical and magnetic charge densities ρ_e and ρ_m we see that (1.1) and (1.2) may be re-expressed in the form,

$$\nabla \times \overline{H} = i\omega\epsilon_0\overline{E} + \overline{J}_e \tag{1.6}$$

where $\overline{J}_e$ is given by

$$\overline{J}_e = i\omega(\overline{\overline{\epsilon}} - \epsilon_0\overline{\overline{I}}) \cdot \overline{E} + \overline{\overline{\sigma}} \cdot \overline{E} \tag{1.7}$$

and the Maxwell equation driven by the magnetic current source term is given by

$$\nabla \times \overline{E} = -i\omega\mu_0\overline{H} + \overline{J}_m \tag{1.8}$$

where the magnetic current density $\overline{J}_m$ is given by

$$\overline{J}_m = i\omega(\overline{\overline{\mu}} - \mu_0\overline{\overline{I}}) \cdot \overline{H} \tag{1.9}$$

The free space isotropic relations yield

$$\nabla \cdot \overline{E} = \frac{\rho_e}{\epsilon_0} \tag{1.10}$$

and

$$\nabla \cdot \overline{H} = \frac{\rho_m}{\mu_0} \tag{1.11}$$

which, respectively, provide us with an operational definition of stimulated electrical and magnetic charge density. Using a pill box concept and the fact that there is no current density in the exterior of the scatterer, we can in addition introduce the notion of surface electric and magnetic charge densities which we denote by η_e and η_m, respectively. The surface electrical charge density is derived through the notion of picking a point on the bounding surface of the scatterer whose normal is $\hat{n}$ and placing a thin volume around this point so that the exterior and interior portions of the boundary have areas equal to A except for a portion of the bounding surface with a very small area whose normal is nearly perpendicular to the normal $\hat{n}$ and so that if $\overline{J}_+$ is the current density at this point just outside the scatterer and $\overline{J}_-$ denotes the current density at this point just inside the scatterer, then conservation of charge on the surface is defined approximately by the relation

$$A\left[\frac{\eta(t+\Delta t)-\eta(t)}{\Delta t}\right] + A\left[(\overline{J}_+ \cdot \hat{n}) - (\overline{J}_- \cdot \hat{n})\right] = 0 \tag{1.12}$$

which means since

$$\overline{J}_+ = 0 \tag{1.13}$$

that the surface charge density, electrical or magnetic, is given by

$$i\omega\eta = \overline{J}_- \cdot \hat{n} \tag{1.14}$$

Thus, the electrical surface charge density is given by equation

$$\eta_e = -\frac{i}{\omega}(\overline{J}_e \cdot \hat{n}) \tag{1.15}$$

and the magnetic surface charge density is given by

$$\eta_m = -i\omega\left[(\overline{\overline{\mu}} - \mu_0\overline{\overline{I}}) \cdot \overline{H}\right] \cdot \hat{n} \tag{1.16}$$

To fully analyze these equations and relate them to the original transmission problem for an anisotropic heterogeneous penetrable scatterer, we need to construct equivalent sources. We will need to consider potentials due to volume and surface electric and magnetic charges and potentials due to volume electric and magnetic currents. From the continuity equation, the electrical charge density satisfies the relation

$$\nabla \cdot \overline{J}_e + \frac{\partial \rho_e}{\partial t} = 0 \tag{1.17}$$

where in our case $\overline{J}_e$ is given by (1.7). A similar relationship for magnetic charge density is developed from a combination of equations (1.11) and (1.9) and the basic Maxwell equation (1.3). The continuity equation for magnetic charge density has exactly the right form in the sense that a valid equation is obtained by replacing e by m in (1.17). By defining electric and magnetic surface charges using the relationships

$$i\omega\eta_e = \left(\left[i\omega(\overline{\overline{\epsilon}} - \epsilon_0\overline{\overline{I}}) + \overline{\overline{\sigma}}\right] \cdot \overline{E}\right) \cdot \hat{n} \tag{1.18}$$

and

$$i\omega\eta_m = \left[-i\omega(\overline{\overline{\mu}} - \mu_0\overline{\overline{I}}) \cdot \overline{H}\right] \cdot \hat{n} \tag{1.19}$$

we complete all we need to derive a coupled set of integral equations in the electric and magnetic field vectors involving both surface and volume integrals. With these definitions we represent the difference between the ambient and total electric vectors in terms of gradients of the potentials of the electric volume and surface charge densities, the electric vector potential, and the curl of the magnetic vector potential. We similarly represent the difference between the total and ambient magnetic field intensity in terms of the gradients of the potentials of the volume and surface magnetic charge densities, the magnetic vector potential, and the curl of the electric vector potential. A coupled electric vector magnetic vector integral equation is immediately derived from the relationships

$$\overline{E} - \overline{E}^i = -\nabla\Phi_e - \nabla\Psi_e - i\omega\overline{A}_e - \frac{1}{\epsilon_0}\nabla \times \overline{A}_m \tag{1.20}$$

and

$$\overline{H} - \overline{H}^i = -\nabla\Phi_m - \nabla\Psi_m - i\omega\overline{A}_m + \frac{1}{\mu_0}\nabla \times \overline{A}_e \tag{1.21}$$

For scatterers with a general shape the set of equations implied by (1.20) and (1.21) are not solvable in closed form. We therefore assume that the scattering body has spherical symmetry and assume a special form of a tensor relationship between $\overline{J}_e$ and $\overline{E}$ and between the magnetic current density $\overline{J}_m$ and the magnetic field intensity $\overline{H}$ in each layer of the structure. This will enable us to get an extension of the usual Mie solution. Specifically we assume that the scattering body Ω consists of N regions delimited by spheres defined by the equations $r = R_i$ for $i = 1, 2, \ldots, N$ where the pth region is bounded by $r = R_p$ and $r = R_{p-1}$ if p is $2, 3, \ldots,$ or N and the core region is bounded by $r = R_1$. In the simplest solution discussed in this paper we assume that we have anisotropic constitutive relations defined in terms of the unit vectors $\hat{e}_r, \hat{e}_\vartheta,$ and $\hat{e}_\phi$, perpendicular to the radial, ϑ, and ϕ coordinate planes, respectively. For time harmonic radiation if $\overline{D}$ is the dielectric displacement and $\overline{J}$ is the ordinary electric field current density, then there are constants ϵ_r and σ_r in the radial direction and constants ϵ and σ for relations along the surface so that

$$\frac{\partial\overline{D}}{\partial t} + \overline{J} = (i\omega\epsilon_r + \sigma_r)E_r\hat{e}_r + (i\omega\epsilon + \sigma)(E_\vartheta\hat{e}_\vartheta + E_\phi\hat{e}_\phi) \tag{1.22}$$

Furthermore, there exist constants related to the magnetic properties of the material denoted by μ_r and μ so that if $\overline{B}$ is the magnetic flux density, and the impinging radiation is time harmonic, then

$$\frac{\partial\overline{B}}{\partial t} = i\omega\mu_r H_r\hat{e}_r + i\omega\mu(H_\vartheta\hat{e}_\vartheta + H_\phi\hat{e}_\phi) \tag{1.23}$$

If we simply require that tangential components of $\overline{E}$ and $\overline{H}$ are continuous across the boundaries $r = R_p$ then expansion coefficients can be related by 2 by 2 matrices as in (Bell, Cohoon, and Penn [1]). If we allow thin impedance sheets between the layers, then the expansion coefficients are also related by 2 by 2 matrices.

Normally when one thinks of scattering by a sphere, the spherical Bessel and Neumann functions come to mind. In our problem we will make use of two Bessel functions with a complex order and will require their evaluation at complex arguments. We assume that ν is complex and define a special function Ψ_ν by the rule

$$\Psi_\nu(z) = \frac{\pi^{1/2} J_{\nu+1/2}(z)}{2^{1/2} z^{1/2}} \tag{1.24}$$

where if we allow W to denote the Bessel function with index equal to $\nu + 1/2$ or specifically

$$W = J_{\nu+\frac{1}{2}}(z) \tag{1.25}$$

this means that the function W satisfies the Bessel differential equation

$$z^2 \frac{d^2 W}{dz^2} + z\frac{dW}{dz} + \left[z^2 - \left(\nu + \frac{1}{2}\right)^2\right] W = 0 \tag{1.26}$$

and then the function Ψ_ν defined by (1.24) through (1.26) satisfies the equation

$$\frac{1}{z}\left(\frac{\partial}{\partial z}\right)^2 (z\Psi_\nu) + \left[1 - \frac{\nu(\nu+1)}{z^2}\right] \Psi_\nu = 0 \tag{1.27}$$

We will need in order to implement our solution on the computer a knowledge of the Wronskian of the independent solutions of the Bessel differential equation (1.26). It is known that

$$-Y_\nu(z)\frac{d}{dz}J_\nu(z) + J_\nu(z)\frac{d}{dz}Y_\nu(z) = \frac{2}{\pi z} \tag{1.28}$$

and this is enough to enable us to evaluate the expressions involving Wronskians of linearly independent solutions of differential equations satisfied by radial functions needed to represent the electric and magnetic fields in the anisotropic materials. We needed two types of differential equations, one with a term involving the ratio of tangential to radial magnetic permeabilities and the other involving a ratio of tangential to radial permittivities. The radial functions associated with a ratio of magnetic properties are denoted by $\Psi^{(a,j)}_{(n,p)}$. The Wronksian determinant, which we denote by Δ is for these functions given by the expression

$$\begin{aligned}\Delta = {} & \Psi^{(a,1)}_{(n,p)}(kR_p)\left(\frac{i}{\omega\mu^{(\nu)}kr}\right)\left[\frac{\partial}{\partial r}\left(r\Psi^{(a,3)}_{(n,p)}(kr)\right)\Big|_{r=R_p}\right] \\ & - \Psi^{(a,3)}_{(n,p)}(kR_p)\left(\frac{i}{\omega\mu^{(\nu)}kr}\right)\left[\frac{\partial}{\partial r}\left(r\Psi^{(a,1)}_{(n,p)}(kr)\right)\Big|_{r=R_p}\right]\end{aligned} \tag{1.29}$$

where the superscript 3 refers to a radial function which is singular at the origin and the superscript 1 refers to a radial function which is regular at $r = 0$ in the integrability sense, and where the radial functions $\Psi^{(a,j)}_{(n,p)}$, are solutions of the differential equation

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\left(r\Psi^{(a,j)}_{(n,p)}\right) + \left[k^2 - \left(\frac{n(n+1)}{r^2}\right)\left(\frac{\omega\mu}{\omega\mu_r}\right)\right]\Psi^{(a,j)}_{(n,p)} = 0 \tag{1.30}$$

where the complex constant k is when r is in the pth layer given by

$$k^2 = \omega^2 \mu^{(p)} \epsilon^{(p)} - i\omega\mu^{(p)}\sigma^{(p)} \tag{1.31}$$

The other radial functions $\Psi^{(b,j)}_{(n,p)}$ satisfy the differential equation

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2 \left(r\Psi^{(b,j)}_{(n,p)}\right) + \left[k^2 - \left(\frac{n(n+1)\zeta}{r^2}\right)\right]\Psi^{(b,j)}_{(n,p)} = 0 \tag{1.32}$$

where

$$\zeta = \left(\frac{i\omega\epsilon^{(p)} + \sigma^{(p)}}{i\omega\epsilon_r^{(p)} + \sigma_r^{(p)}}\right) \tag{1.33}$$

We remark that the entire theory could be made completely symmetric by introducing a magnetic charge conductivity so that the Maxwell equations would have the form

$$\nabla \times \overline{E} = -i\omega\mu_0\overline{H} - \left[i\omega(\overline{\overline{\mu}} - \mu_0\overline{\overline{I}}) + \overline{\overline{\sigma}}_m\right] \cdot \overline{H} \tag{1.34}$$

$$\nabla \times \overline{H} = i\omega\epsilon_0\overline{E} + \left[i\omega(\overline{\overline{\epsilon}} - \epsilon_0\overline{\overline{I}}) + \overline{\overline{\sigma}}_e\right] \cdot \overline{E} \tag{1.35}$$

With these formulations of the Maxwell equations, we can use the potential representation of the electric and magnetic field vectors to write a surface volume integral equation system coupling the electric and magnetic vectors in the form, described by Graglia and Uslenghi [8] in their paper on electromagnetic scattering by anisotropic materials with a completely general shape.

As a part of this formulation we make use of the temperate rotationally invariant fundamental solution of the Helmholtz operator of free space derived in Treves [31] using the Haar measure on the rotation group, given by

$$G(r,s) = \frac{\exp(-ik_0|r-s|)}{4\pi|r-s|} \tag{1.36}$$

The coupled electric vector magnetic vector integral equations describing the interaction of radiation with an anisotropic material are given by

$$\begin{aligned}
\overline{E} - \overline{E}^i = &- \nabla \int_\Omega \frac{i\nabla \cdot \left[i\omega(\overline{\overline{\epsilon}} - \epsilon_0\overline{\overline{I}}) + \overline{\overline{\sigma}}_e\right] \cdot \overline{E}}{\omega\epsilon_0}(s)\, G(r,s)\, dv(s) \\
&+ \frac{i}{\omega\epsilon_0}\nabla \int_{\partial\Omega} \left(\left[i\omega(\overline{\overline{\epsilon}} - \epsilon_0\overline{\overline{I}} + \overline{\overline{\sigma}}_e) \cdot \overline{E}\right] \cdot \hat{n}\right)(s)\, G(r,s)\, da(s) \\
&- i\omega\mu_0 \int_\Omega \left[i\omega(\overline{\overline{\epsilon}} - \epsilon_0\overline{\overline{I}}) + \overline{\overline{\sigma}}_e\right] \cdot \overline{E}(s)\, G(r,s)\, dv(s) \\
&- \nabla \times \int_\Omega \left[i\omega(\overline{\overline{\mu}} - \mu_0\overline{\overline{I}}) + \overline{\overline{\sigma}}_m\right] \cdot \overline{H}(s)\, G(r,s)\, dv(s)
\end{aligned} \tag{1.37}$$

and

$$
\begin{aligned}
\overline{H} - \overline{H}_i = & - \nabla \int_{\Omega} \frac{i\nabla \cdot \left[i\omega(\overline{\overline{\mu}} - \mu_0 \overline{\overline{I}}) + \overline{\overline{\sigma}}_m \right] \cdot \overline{H}}{\omega\mu_0} (s)\, G(r,s)\, dv(s) \\
& - \frac{i}{\omega\mu_0} \nabla \int_{\partial\Omega} \left(\left[i\omega(\overline{\overline{\mu}} - \mu_0 \overline{\overline{I}}) + \overline{\overline{\sigma}}_m \right] \cdot \overline{H} \cdot \hat{n} \right)(s)\, G(r,s)\, da(s) \\
& - i\omega\epsilon_0 \int_{\Omega} \left[i\omega(\overline{\overline{\mu}} - \mu_0 \overline{\overline{I}}) + \overline{\overline{\sigma}}_m \right] \cdot \overline{H}(s)\, G(r,s)\, dv(s) \\
& + \nabla \times \int_{\Omega} \left[i\omega \left(\overline{\overline{\epsilon}} - \epsilon_0 \overline{\overline{I}} \right) + \overline{\overline{\sigma}}_e \right] \cdot \overline{E}(s)\, G(r,s)\, dv(s) \qquad (1.38)
\end{aligned}
$$

We can make use of the energy balance relationship described in [1] to express the total absorbed power in terms of coefficients used to express the radiation scattered away from the sphere and the expansion coefficients of the incident radiation. This gives us the extinction and scattering cross sections of our spherical structure. The same results are necessarily obtained by integration of the power density distribution over the interior of the sphere. This has been done for a variety of anisotropic structures as a check on our computer program. The triple integral and the formula in Bell et al. [1] give answers agreeing to at least 7 decimal places when 12-point Gauss quadrature is used for integrating with respect to θ, ϕ, and r in each layer of the sphere. The agreement persists even when different ratios of real and imaginary parts of radial and tangential permittivity, and conductivity are used. The possibility of a purely anisotropic loss can be seen by observing the energy conservation relation (c.f. Kong [18]) for a general anisotropic material. For isotropic materials the situation is different in that, for example, to have a magnetic loss in a linearly responding material one must have a magnetic permeability with a nontrivial imaginary part.

For an anisotropic material with a completely general shape the problem of computing the interaction at first glance may seem formidable. However, by the use of the method of Cohoon [3] and the resolvent kernel methods presented by this author at the Midwest Conference on Differential Equations held at Vanderbilt University in Nashville, Tennessee on October 23–24, 1987, one can develop a robust method of solving the coupled system, given by (1.37) and (1.38), that does not require excessive computer memory.

2. REPRESENTATION OF THE FIELDS

We attempt to develop an electric vector representing a solution of the Maxwell equations in spherical coordinates in the interior of an anisotropic body. *A priori* we consider in the pth layer of the sphere three radial functions which we denote by $\varphi^{(a,j)}_{(n,p)}$, $\varphi^{(b,j)}_{(n,p)}$ and $\varphi^{(c,j)}_{(n,p)}$, and we assume furthermore that within this layer the electric vector $\overline{E}$ of a solution of the Maxwell equations has the form

$$
\begin{aligned}
\overline{E} = \sum_{(m,n)\in I} \Bigg\{ & e_{(m,n)} \Psi^{(a,j)}_{(n,p)} \left[im \frac{P_n^m(\cos\theta)}{\sin\theta}\,\hat{e}_\theta - \frac{d}{d\theta} P_n^m(\cos\theta)\,\hat{e}_\phi \right] e^{im\phi} \\
& + c_{(m,n)} \frac{\Psi^{(c,j)}_{(n,p)}(kr)}{kr} P_n^m(\cos\theta) e^{im\phi}\,\hat{e}_r \\
& + b_{(m,n)} \left(-\frac{1}{kr} \right) \frac{\partial}{\partial r} \left[r \Psi^{(b,j)}_{(n,p)}(kr) \right] \left[\frac{d}{d\theta} P_n^m(\cos\theta)\,\hat{e}_\theta \right. \\
& \left. + im \frac{P_n^m(\cos\theta)}{\sin\theta}\,\hat{e}_\theta \right] e^{im\phi} \Bigg\}
\end{aligned}
\tag{2.1}
$$

We will suppose that the material within the layers satisfies the constitutive relations described in the Introduction. We define singular vector fields on the sphere by the rules

$$\overline{A}_{(m,n)}(\theta,\phi) = \left[im \frac{P_n^m(\cos\theta)}{\sin\theta}\,\hat{e}_\theta - \frac{d}{d\theta} P_n^m(\cos\theta)\,\hat{e}_\phi \right] e^{im\phi} \tag{2.2}$$

$$\overline{C}_{(m,n)}(\theta,\phi) = P_n^m(\cos\theta) e^{im\phi}\,\hat{e}_r \tag{2.3}$$

and

$$\overline{B}_{(m,n)}(\theta,\phi) = \left[\frac{d}{d\theta} P_n^m(\cos\theta)\,\hat{e}_\theta + im \frac{P_n^m(\cos\theta)}{\sin\theta}\,\hat{e}_\phi \right] e^{im\phi} \tag{2.4}$$

We now make use of the following Lemma to simplify the computation of the curl of vector fields and especially $\nabla \times \overline{H}$.

Lemma 2.1. If $F(r)$ is a differentiable function of r, and if

$$\overline{A} = F(r)\,\overline{A}_{(m,n)}(\theta,\phi)$$

or, equivalently

$$F(r)\,\overline{A}_{(m,n)}(\theta,\phi) = F(r) \left[im \frac{P_n^m(\cos\theta)}{\sin\theta}\,\hat{e}_\theta - \frac{d}{d\theta} P_n^m(\cos\theta)\,\hat{e}_\phi \right] e^{im\phi} \tag{2.5}$$

$$\overline{C} = F(r)\,\overline{C}_{(m,n)}(\theta,\phi) = F(r)\,P_n^m(\cos\theta)\,e^{im\phi}\,\hat{e}_r \tag{2.6}$$

and

$$
\begin{aligned}
\overline{B} &= F(r)\,\overline{B}_{(m,n)}(\theta,\phi) \\
&= F(r) \left[\frac{d}{d\theta} P_n^m(\cos\theta)\,\hat{e}_\theta + im \frac{P_n^m(\cos\theta)}{\sin\theta}\,\hat{e}_\phi \right] e^{im\phi}
\end{aligned}
\tag{2.7}
$$

then

$$\nabla \times \overline{A} = n(n+1) \frac{F(r)}{r}\,\overline{C}_{(m,n)}(\theta,\phi) + \left[\frac{1}{r} \frac{\partial}{\partial r} \Big(rF(r) \Big) \right] \overline{B}_{(m,n)}(\theta,\phi) \tag{2.8}$$

$$\nabla \times \overline{C} = \frac{F(r)}{r}\,\overline{A}_{(m,n)}(\theta,\phi) \tag{2.9}$$

and

$$\nabla \times \overline{B} = \left[-\frac{1}{r} \frac{\partial}{\partial r} \Big(rF(r) \Big) \right] \overline{A}_{(m,n)}(\theta,\phi) \tag{2.10}$$

Proof. This follows from the fact that in spherical coordinates if

$$\overline{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_\phi \hat{e}_\phi \tag{2.11}$$

then

$$\begin{aligned}\nabla \times \overline{v} = & \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{e}_r \\ & + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{e}_\theta \\ & + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{e}_\phi\end{aligned} \tag{2.12}$$

and the relation

$$\begin{aligned}-\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d}{d\theta}P_n^m(\cos\theta)\right] + \frac{m^2}{\sin^2\theta}P_n^m(\cos\theta) \\ = n(n+1)P_n^m(\cos\theta)\end{aligned} \tag{2.13}$$

which is simply the Legendre differential equation which is usually expressed in the form

$$\frac{d}{dz}(1-z^2)\frac{dW}{dz} + \left[n(n+1) - \frac{m^2}{1-z^2}\right]W = 0 \tag{2.14}$$

where $z = \cos\theta$.

We now use the definition of $\overline{E}$, (2.1), the definition of the three sections, (2.2)-(2.4), in the tangent and the normal bundle of the sphere, and Lemma 2.1 to represent $\overline{E}$ as

$$\begin{aligned}\overline{E} = \sum_{(m,n)\in I}\Big\{ & a_{(m,n)} Z_n^{(a)}(r)\,\overline{A}_{(m,n)}(\theta,\phi) + c_{(m,n)}\frac{Z_n^{(c)}(r)}{kr}\,\overline{C}_{(m,n)}(\theta,\phi) \\ & + b_{(m,n)}\left[\frac{-1}{kr}\frac{\partial}{\partial r}\left(r Z_n^{(b)}(r)\right)\right]\overline{B}_{(m,n)}(\theta,\phi)\Big\}\end{aligned} \tag{2.15}$$

where

$$Z_n^{(a)}(r) = \Psi_{(n,p)}^{(a,j)}(kr) \tag{2.16}$$

$$Z_n^{(b)}(r) = \Psi_{(n,p)}^{(b,j)}(kr) \tag{2.17}$$

$$Z_n^{(c)}(r) = \Psi_{(n,p)}^{(c,j)}(kr) \tag{2.18}$$

where the functions on the right side of (2.16) through (2.18) are defined by (1.30) through (1.33) but may actually be conceptualized at this stage of development as generalizations of these solutions. In developing the solution of the Maxwell equations we have to compute the curl of the vector field $\overline{E}$. We find that

$$\nabla \times \overline{E} = \sum_{(m,n)\in I}\Big\{ a_{(m,n)}\frac{Z_n^{(a)}(r)\, n(n+1)}{r}\,\overline{C}_{(m,n)}(\theta,\phi)$$

$$+ a_{(m,n)}\frac{1}{r}\frac{\partial}{\partial r}\left[rZ_n^{(a)}(r)\right]\overline{B}_{(m,n)}(\theta,\phi) + c_{(m,n)}\frac{Z_n^{(c)}(r)}{kr^2}\overline{A}_{(m,n)}(\theta,\phi)$$
$$+\frac{b_{(m,n)}}{kr}\left(\frac{\partial}{\partial r}\right)^2\left[rZ_n^{(b)}(r)\right]\overline{A}_{(m,n)}(\theta,\phi)\Bigg\} \tag{2.19}$$

Equation (2.19), the Maxwell equation (1.2) and (1.23) therefore imply that the magnetic field intensity is given by

$$\overline{H} = \sum_{(m,n)\in I}\left\{\frac{i}{\omega\mu_r}a_{(m,n)}\frac{Z_n^{(a)}(r)\,n(n+1)}{r}\overline{C}_{(m,n)}(\theta,\phi)\right.$$
$$+ a_{(m,n)}\frac{i}{\omega\mu}\frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(r)\right)\overline{B}_{(m,n)}(\theta,\phi)$$
$$\left.+\frac{i}{\omega\mu}\left[c_{(m,n)}\frac{Z_n^{(c)}(r)}{kr^2} + b_{(m,n)}\frac{1}{k_r}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(r)\right)\right]\overline{A}_{(m,n)}(\theta,\phi)\right\} \tag{2.20}$$

We now will get the final Maxwell equation relating the curl of the magnetic field intensity to the electric field vector through a tensor relationship. We find that (2.2)–(2.10) and (2.20) and the Maxwell equation (1.1) and the original representation, (2.1), of $\overline{E}$ imply that

$$\nabla\times\overline{H} = \sum_{(m,n)\in I}\Bigg\{$$
$$a_{(m,n)}\left[\frac{i}{\omega\mu_r}\left(\frac{Z_n^{(a)}(r)\,n(n+1)}{r^2}\right) + \frac{i}{\omega\mu}\left(-\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(a)}(r)\right)\right)\right]\overline{A}_{(m,n)}(\theta,\phi)$$
$$+\frac{n(n+1)}{r}\frac{i}{\omega\mu}\left[c_{(m,n)}\frac{Z_n^{(c)}(r)}{kr^2} + b_{(m,n)}\frac{1}{kr}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(r)\right)\right]\overline{C}_{(m,n)}(\theta,\phi)$$
$$+\frac{1}{k_pr}\frac{i}{\omega\mu}\frac{\partial}{\partial r}\left[c_{(m,n)}\frac{Z_n^{(c)}(kr)}{r} + b_{(m,n)}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(r)\right)\right]\overline{B}_{(m,n)}(\theta,\phi)\Bigg\}$$
$$= \sum_{(m,n)\in I}\Bigg\{(i\omega\epsilon + \sigma)\,a_{(m,n)}Z_n^{(a)}(r)\,\overline{A}_{(m,n)}(\theta,\phi)$$
$$+ (i\omega\epsilon_r + \sigma_r)c_{(m,n)}\frac{Z_n^{(c)}(r)}{kr}\overline{C}_{(m,n)}(\theta,\phi)$$
$$+ (i\omega\epsilon + \sigma)b_{(m,n)}\left[-\frac{1}{kr}\frac{\partial}{\partial r}\left(rZ_n^{(b)}(r)\right)\right]\overline{B}_{(m,n)}(\theta,\phi)\Bigg\} \tag{2.21}$$

The first differential equation that we derive is obtained by equating coefficients of the vector field $\overline{A}_{(m,n)}$ on both sides of (2.21) and is given by

$$\frac{iZ_n^{(a)}(r)\,n(n+1)}{r^2\omega\mu_r} + \left(-\frac{i}{\omega\mu}\right)\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(a)}(r)\right) = (i\omega\epsilon + \sigma)Z_n^{(a)}(r) \tag{2.22}$$

We can do this because of the orthogonality relationships

$$\int_0^{2\pi}\int_0^{\pi}\overline{A}_{(m,n)}(\theta,\phi)\cdot\overline{B}_{(m,n)}(\theta,\phi)\sin\theta\,d\theta\,d\phi=0 \tag{2.23}$$

and

$$\overline{A}_{(m,n)}\cdot\overline{C}_{(m,n)}=0=\overline{B}_{(m,n)}\cdot\overline{C}_{(m,n)} \tag{2.24}$$

Notice that the differential equation may be rewritten in the form

$$-\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(a)}(r)\right)+Z_n^{(a)}(r)\left[\frac{n(n+1)\,\omega\mu}{r^2\omega\mu_r}\right]=k^2Z_n^{(a)}(r) \tag{2.25}$$

where

$$k^2=\omega^2\mu\epsilon-i\omega\mu\sigma \tag{2.26}$$

We note that the ordinary spherical Bessel function satisfies the relationship

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\left[rj_n(kr)\right]+\left[k^2-\frac{n(n+1)}{r^2}\right]j_n(kr)=0 \tag{2.27}$$

which shows that when the two permeabilities approach one another, the radial function becomes simply a spherical Bessel function with an integer index.

The next radial differential equation can be obtained by equating coefficients of $\hat{e}_r$ on both sides of (2.21) and by making use of the properties of the traditional scalar spherical harmonics $P_n^m(\cos\theta)$ (Bell, Cohoon, and Penn [2]). These considerations give us the relationship

$$\begin{aligned}(i\omega\epsilon_r+\sigma_r)\,c_{(m,n)}\frac{Z_n^{(c)}(r)}{r}&\\ =\frac{in(n+1)}{\omega\mu r}&\left[c_{(m,n)}\frac{Z_n^{(c)}(r)}{r^2}+b_{(m,n)}\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(r)\right)\right]\end{aligned} \tag{2.28}$$

The final radial differential equation is obtained by equating coefficients of the vector field, $\overline{B}_{(m,n)}(\theta,\phi)$ defined by (2.4) on both sides of (2.21). This differential equation has the form

$$\begin{aligned}(i\omega\epsilon+\sigma)\,b_{(m,n)}\left(-\frac{1}{r}\right)\frac{\partial}{\partial r}\left(rZ_n^{(b)}(r)\right)&\\ =\frac{1}{r}\frac{\partial}{\partial r}\left\{\frac{i}{\omega\mu}\left[c_{(m,n)}\frac{Z_n^{(c)}(r)}{r}+b_{(m,n)}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(r)\right)\right]\right\}&\end{aligned} \tag{2.29}$$

We have consistency between the two differential equations, (2.28) and (2.29), if

$$\frac{(i\omega\epsilon_r+\sigma_r)\,c_{(m,n)}}{n(n+1)\,r}\frac{\partial}{\partial r}\left(rZ_n^{(c)}(r)\right)=-b_{(m,n)}(i\omega\epsilon+\sigma)\frac{1}{r}\frac{\partial}{\partial r}rZ_n^{(b)}(r) \tag{2.30}$$

We find that we get a very simple solution of these equations if we simply let

$$Z_n^{(b)}=Z_n^{(c)}(r) \tag{2.31}$$

and assume that

$$c_{(m,n)} = \frac{-n(n+1)(i\omega\epsilon + \sigma)}{(i\omega\epsilon_r + \sigma_r)} b_{(m,n)} \tag{2.32}$$

Under our simple hypothesis we derive two distinct radial differential equations. As in the traditional Mie solution (Bell, Cohoon, and Penn [1]) we have multipliers of the $\overline{A}$ vector fields which are of the same type that one gets by computing the curl of the product of a solution of the scalar Helmholtz equation by the unit vector $\hat{e}_r$ and vector fields which have the same form as the curl of a vector field of this type. In the traditional solution [1] the coefficients multiplying the terms involving regular and singular radial functions times the first type of vector field are labeled with a and α, respectively, and the coefficients multiplying the terms involving the regular and singular functions times the second type of vector field are labeled with b and β, respectively.

Combining (2.28) and (2.29) and making use of the assumptions embodied in (2.30)–(2.32) give us the differential equation

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(b)}(r)\right) + \left[k^2 - \frac{n(n+1)}{r^2}\left(\frac{i\omega\epsilon + \sigma}{i\omega\epsilon_r + \sigma_r}\right)\right] Z_n^{(b)}(r) = 0 \tag{2.33}$$

Making use of this second radial differential equation, and making use of the relationships between the coefficients, equations (2.30) through (2.32), we will get a new representation of the magnetic field intensity $\overline{H}$ that was originally given by (2.20). Specifically, we need to first look at the term involving the $\overline{A}$ vector in (2.20). The relevant observation, using (2.28), is that

$$\begin{aligned}
&\left[c_{(m,n)}\frac{Z_n^{(b)}(r)}{r^2} + b_{(m,n)}\frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(c)}(r)\right)\right] \\
&= \left[\frac{-n(n+1)(i\omega\epsilon + \sigma)}{i\omega\epsilon_r + \sigma_r}\left(\frac{Z_n^{(c)}(r)}{r^2}\right) + \frac{1}{r}\left(\frac{\partial}{\partial r}\right)^2\left(rZ_n^{(c)}(r)\right)\right] b_{(m,n)} \\
&= -i\omega\mu(i\omega\epsilon_r + \sigma_r)\left[\frac{Z_n^{(c)}(r)}{n(n+1)}\right] c_{(m,n)} \\
&= -k^2 Z_n^{(c)}(r)\, b_{(m,n)}
\end{aligned} \tag{2.34}$$

Combining these equations we obtain a greatly simplified expression for $\overline{H}$ of the form

$$\begin{aligned}
\overline{H} = \sum_{(m,n)\in I} &\left\{\frac{i}{\omega\mu_r} a_{(m,n)} Z_n^{(a)}(r)\frac{n(n+1)}{r}\,\overline{C}_{(m,n)}(\theta,\phi)\right. \\
&+ \left[\frac{i}{\omega\mu} a_{(m,n)}\frac{1}{r}\frac{\partial}{\partial r}\left(rZ_n^{(a)}(r)\right)\right]\overline{B}_{(m,n)}(\theta,\phi) \\
&\left.+ \frac{i}{\omega\mu}(-k)Z_n^{(b)}(r)\, b_{(m,n)}\overline{A}_{(m,n)}(\theta,\phi)\right\}
\end{aligned} \tag{2.35}$$

The simple relationships (2.15) and (2.35) give us easy matrix relationships between expansion coefficients used to represent the field in one layer to those

used to represent the field in another layer. Details concerning these intralayer relationships are discussed in the next sections of the paper. We will discuss spherical structures with a metallic core, dielectric multilayers where the layers may have nontrivial magnetic properties, and structures where the layers are separated by charge sheets or very thin layers, referred to in the literature, as impedance sheets.

3. INTRALAYER RELATIONSHIPS

The purpose of this section is to develop matrix equations which relate the expansion coefficients in one layer to those in adjacent layers and ultimately to be able to express the expansion coefficients of the field in any layer to the expansion coefficients of the incident radiation. The program will treat both structures with metallic cores and dielectric multilayers with nontrivial magnetic properties. We will choose four expansion coefficients for each layer. These expansion coefficients will be $a^{(p)}_{(m,n)}$, $b^{(p)}_{(m,n)}$, $\alpha^{(p)}_{(m,n)}$ and $\beta^{(p)}_{(m,n)}$.

The electric vector in layer p is given by

$$\begin{aligned}\overline{E} = \sum_{(m,n)\in I} \Bigg\{ & a^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p)}(r)\, \overline{A}_{(m,n)}(\theta,\phi) \\ & + b^{(p)}_{(m,n)} \left(-\frac{1}{kr}\right) \frac{\partial}{\partial r}\left(r Z^{(b,1)}_{(n,p)}(r)\right) \overline{B}_{(m,n)}(\theta,\phi) \\ & + \left[\frac{-n(n+1)(i\omega\epsilon^{(p)} + \sigma^{(p)})}{i\omega\epsilon^{(p)}_r + \sigma^{(p)}_r}\right] b^{(p)}_{(m,n)} \frac{Z^{(b,1)}_{(n,p)}(r)}{kr} \overline{C}_{(m,n)}(\theta,\phi) \\ & + \alpha^{(p)}_{(m,n)} Z^{(a,3)}_{(n,p)}(r)\, \overline{A}_{(m,n)}(\theta,\phi) + \beta^{(p)}_{(m,n)} \left(-\frac{1}{kr}\right) \frac{\partial}{\partial r}\left(r Z^{(b,3)}_{(n,p)}(r)\right) \overline{B}_{(m,n)}(\theta,\phi) \\ & + \left[\frac{-n(n+1)\left(i\omega\epsilon^{(p)} + \sigma^{(p)}\right)}{i\omega\epsilon^{(p)}_r + \sigma^{(p)}_r}\right] \beta^{(p)}_{(m,n)} \frac{Z^{(b,3)}_{(n,p)}(r)}{kr} \overline{C}_{(m,n)}(\theta,\phi) \Bigg\} \qquad (3.1)\end{aligned}$$

It is now very easy in view of the relationships given in Lemma 2.1, where the vectors $\overline{A}_{(m,n)}$, $\overline{B}_{(m,n)}$, and $\overline{C}_{(m,n)}$, are given by (2.2), (2.3), and (2.4), respectively, to calculate the magnetic field intensity $\overline{H}$ and show that in view of orthogonality relationships (2.23), (2.24), and the additional relationships, valid when the index q is different from n which state that

$$\int_0^{2\pi}\int_0^{\pi} \overline{A}_{(m,n)}(\theta,\phi)\cdot\overline{A}_{(m,q)}(\theta,\phi)\sin\theta\, d\theta\, d\phi = 0$$

and that

$$\int_0^{2\pi}\int_0^{\pi} \overline{B}_{(m,n)}(\theta,\phi)\cdot\overline{B}_{(m,q)}(\theta,\phi)\sin\theta\, d\theta\, d\phi = 0$$

that very simple relationships result when we equate tangential components of $\overline{E}$ and $\overline{H}$ across the boundaries.

The magnetic intensity vector $\overline{H}$ is given by

$$\overline{H} = \sum_{(m,n)\in I} \left\{ \frac{i}{\omega\mu^{(p)}} \left[-k_p b_{(m,n)}^{(p)} Z_{(n,p)}^{(b,1)}(r)\, \overline{A}_{(m,n)}(\theta,\phi) \right.\right.$$
$$\left. + a_{(m,n)}^{(p)} \frac{1}{r}\frac{\partial}{\partial r}\left(r Z_{(n,p)}^{(a,1)}(r)\right) \overline{B}_{(m,n)}(\theta,\phi) \right]$$
$$+ \frac{i}{\omega\mu_r^{(p)}} a_{(m,n)}^{(p)} Z_{(n,p)}^{(a,1)}(r) \frac{n(n+1)}{r} \overline{C}_{(m,n)}(\theta,\phi)$$
$$+ \frac{i}{\omega\mu^{(p)}} \left[-k_p \beta_{(m,n)}^{(p)} Z_{(n,p)}^{(b,3)}(r) \overline{A}_{(m,n)}(\theta,\phi) \right.$$
$$\left. + \alpha_{(m,n)}^{(p)} \frac{1}{r}\frac{\partial}{\partial r}\left(r Z_{(n,p)}^{(a,3)}(r)\right) \overline{B}_{(m,n)}(\theta,\phi) \right]$$
$$\left. + \frac{i}{\omega\mu_r} \alpha_{(m,n)}^{p} Z_{(n,p)}^{(a,3)}(r) \frac{n(n+1)}{r} \overline{C}_{(m,n)}(\theta,\phi) \right\} \tag{3.2}$$

The ordinary boundary value problem requires continuity of tangential components of $\overline{E}$ and $\overline{H}$ across the boundary layers

$$r = R_p \tag{3.3}$$

Making use of this and the orthogonality relationships we deduce that if we let

$$W_{(n,p)}^{(a,j)}(R_p) = \frac{1}{k_p r}\frac{\partial}{\partial r}\left(r Z_{(n,p)}^{(a,j)}(r)\right) \Big|_{r=R_p} \tag{3.4}$$

and

$$W_{(n,p+1)}^{(a,j)}(R_p) = \left(\frac{1}{k_{p+1} r}\right) \frac{\partial}{\partial r}\left(r Z_{(n,p+1)}^{(a,j)}(r)\right) \Big|_{r=R_p} \tag{3.5}$$

and if we let

$$\rho_p = \frac{\mu^{(p)} k_{p+1}}{\mu^{(p+1)} k_p} \tag{3.6}$$

then the expansion coefficients are related by a matrix equation

$$\begin{pmatrix} Z_{(n,p)}^{(a,1)}(R_p) & Z_{(n,p)}^{(a,3)}(R_p) \\ W_{(n,p)}^{(a,1)}(R_p) & W_{(n,p)}^{(a,3)}(R_p) \end{pmatrix} \begin{pmatrix} a_{(m,n)}^{(p)} \\ \alpha_{(m,n)}^{(p)} \end{pmatrix} =$$
$$\begin{pmatrix} Z_{(n,p+1)}^{(a,1)}(R_p) & Z_{(n,p+1)}^{(a,3)}(R_p) \\ \rho_p W_{(n,p+1)}^{(a,1)}(R_p) & \rho_p W_{(n,p+1)}^{(a,3)}(R_p) \end{pmatrix} \begin{pmatrix} a_{(m,n)}^{(p+1)} \\ \alpha_{(m,n)}^{(p+1)} \end{pmatrix} \tag{3.7}$$

The matrix relationship (3.7) may be abbreviated as

$$\overline{\overline{S}}^{(p)}(R_p) \begin{pmatrix} a_{(m,n)}^{(p)} \\ \alpha_{(m,n)}^{(p)} \end{pmatrix} = \overline{\overline{S}}^{(p+1)}(R_p) \begin{pmatrix} a_{(m,n)}^{(p+1)} \\ \alpha_{(m,n)}^{(p+1)} \end{pmatrix} \tag{3.8}$$

We obtained the matrix relationship (3.7) by equating tangential components of $\overline{E}$ and $\overline{H}$ across the boundary of the spherical surface separating the layers, taking the inner product of both sides of this equation with respect to $\overline{A}_{(m,n)}$ and then integrating over the surface of this sphere.

A second matrix relationship is obtained by equating tangential components of $\overline{E}$ and $\overline{H}$ across the separating spherical boundary, taking the inner product of both sides of these equations with respect to the vector functions $\overline{B}_{(m,n)}$, and then integrating over the surface of the sphere; this second matrix equation relates the expansion coefficients $b^{(p)}_{(m,n)}$ and $\beta^{(p)}_{(m,n)}$ to those in layer $p+1$, and is given by

$$\begin{pmatrix} Z^{(b,1)}_{(n,p)}(R_p) & Z^{(b,3)}_{(n,p)}(R_p) \\ W^{(b,1)}_{(n,p)}(R_p) & W^{(b,3)}_{(n,p)}(R_p) \end{pmatrix} \begin{pmatrix} b^{(p)}_{(m,n)} \\ \beta^{(p)}_{(m,n)} \end{pmatrix} = \begin{pmatrix} \rho_p Z^{(b,1)}_{(n,p+1)}(R_p) & \rho_p Z^{(b,3)}_{(n,p+1)}(R_p) \\ W^{(b,1)}_{(n,p+1)}(R_p) & W^{(b,3)}_{(n,p+1)}(R_p) \end{pmatrix} \begin{pmatrix} b^{(p+1)}_{(m,n)} \\ \beta^{(p+1)}_{(m,n)} \end{pmatrix} \tag{3.9}$$

The matrix relationship (3.9) may be rewritten in abridged form as

$$\overline{\overline{T}}^{(p)}(R_p) \begin{pmatrix} b^{(p)}_{(m,n)} \\ \beta^{(p)}_{(m,n)} \end{pmatrix} = \overline{\overline{T}}^{(p+1)}(R_p) \begin{pmatrix} b^{(p+1)}_{(m,n)} \\ \beta^{(p+1)}_{(m,n)} \end{pmatrix} \tag{3.10}$$

Define new matrices by the rules

$$\overline{\overline{Q}}^{(p)}_{(m,n)} = \overline{\overline{S}}^{(p)-1}_{(m,n)}(R_p)\, \overline{\overline{S}}^{(p-1)}_{(m,n)}(R_p) \tag{3.11}$$

and

$$\overline{\overline{R}}^{(p)}_{(m,n)} = \overline{\overline{T}}^{(p)-1}_{(m,n)}(R_p)\, \overline{\overline{T}}^{(p+1)}_{(m,n)}(R_p) \tag{3.12}$$

Thus, if region $N+1$ is the region surrounding the sphere, we assume that $a^{(N+1)}_{(m,n)}$ and $b^{(N+1)}_{(m,n)}$ are all completely known. Thus,

$$\alpha^{(1)}_{(m,n)} = \beta^{(1)}_{(m,n)} = 0 \tag{3.13}$$

for all nonnegative integers n and all integers m not smaller than $-n$ nor larger than $+n$. Thus, we see that as in the case of isotropic N-layer spherical structures (Bell, Cohoon, and Penn [1]) we have the relationships

$$\begin{pmatrix} a^{(1)}_{(m,n)} \\ 0 \end{pmatrix} = \overline{\overline{Q}}^{(m,n)} \begin{pmatrix} a^{(N+1)}_{(m,n)} \\ \alpha^{(N+1)}_{(m,n)} \end{pmatrix} \tag{3.14}$$

where the matrix $\overline{\overline{Q}}^{(m,n)}$ is given by

$$\overline{\overline{Q}}^{(m,n)} = \overline{\overline{Q}}^{(1)}_{(m,n)}(R_1)\, \overline{\overline{Q}}^{(2)}_{(m,n)}(R_2) \ldots \overline{\overline{Q}}^{(N)}_{(m,n)}(R_N) \tag{3.15}$$

The matrix relationship (3.14) yields two equations in two unknowns which in turn imply that

$$a^{(1)}_{(m,n)} = Q^{(m,n)}_{(1,1)} a^{(N+1)}_{(m,n)} + Q^{(m,n)}_{(1,2)} \alpha^{(N+1)}_{(m,n)} \tag{3.16}$$

and

$$0 = Q^{(m,n)}_{(2,1)} a^{(N+1)}_{(m,n)} + Q^{(m,n)}_{(2,2)} \alpha^{(N+1)}_{(m,n)} \tag{3.17}$$

Thus, we see that

$$\alpha^{(N+1)}_{(m,n)} = \frac{-Q^{(m,n)}_{(2,1)} a^{(N+1)}_{(m,n)}}{Q^{(m,n)}_{(2,2)}} \tag{3.18}$$

and

$$a^{(1)}_{(m,n)} = \left[Q^{(m,n)}_{(1,1)} + Q^{(m,n)}_{(1,2)} \left(\frac{-Q^{(m,n)}_{(2,1)}}{Q^{(m,n)}_{(2,2)}} \right) \right] a^{(N+1)}_{(m,n)} \tag{3.19}$$

Similarly, we see that

$$\beta^{(N+1)}_{(m,n)} = \frac{-R^{(m,n)}_{(2,1)} b^{(N+1)}_{(m,n)}}{R^{(m,n)}_{(2,2)}} \tag{3.20}$$

and

$$b^{(1)}_{(m,n)} = \left[R^{(m,n)}_{(1,1)} + R^{(m,n)}_{(1,2)} \left(\frac{-R^{(m,n)}_{(2,1)}}{R^{(m,n)}_{(2,2)}} \right) \right] b^{(N+1)}_{(m,n)} \tag{3.21}$$

where

$$\overline{\overline{R}}^{(m,n)} = \overline{\overline{R}}^{(1)}_{(m,n)}(R_1)\, \overline{\overline{R}}^{(2)}_{(m,n)}(R_2) \ldots \overline{\overline{R}}^{(N)}_{(m,n)}(R_N) \tag{3.22}$$

From these relationships the expansion coefficients in all layers can be determined from the expansion coefficients in the inner core of the multilayer structure. The basic relationships are

$$\begin{pmatrix} a^{(2)}_{(m,n)} \\ c^{(2)}_{(m,n)} \end{pmatrix} = \left(\overline{\overline{Q}}^{(1)}_{(m,n)} \right)^{-1} \begin{pmatrix} a^{(1)}_{(m,n)} \\ 0 \end{pmatrix} \tag{3.23}$$

and

$$\begin{pmatrix} b^{(2)}_{(m,n)} \\ \beta^{(2)}_{(m,n)} \end{pmatrix} = \left(\overline{\overline{R}}^{(1)}_{(m,n)} \right)^{-1} \begin{pmatrix} b^{(1)}_{(m,n)} \\ 0 \end{pmatrix} \tag{3.24}$$

From (3.23) and (3.24) we see that all the coefficients in layer 2 are now completely known. By iterating these results we can get all the coefficients in layers 3 through N. Since we already know the expansion coefficients in layers 1 and 2 and in the region of free space surrounding the N-layer structure we see that by making use of the formulas (3.1) and (3.2) we get the components of the electric and magnetic field vectors in each layer of the structure and in the region of free space surrounding the structure.

4. INTRALAYER RELATIONSHIPS WITH A PERFECTLY CONDUCTING CORE

Let us suppose that the inner core of an N-layer structure is perfectly conducting which means that on this inner layer defined in spherical coordinates by

$$r = R_1 \tag{4.1}$$

the electric vector $\overline{E}$ defined by (3.1) is identically zero which means, in view of the orthogonality relationships described in Section 3, that

$$a^{(2)}_{(m,n)} Z^{(a,1)}_{(n,2)}(R_1) + \alpha^{(2)}_{(m,n)} Z^{(a,3)}_{(n,2)}(R_1) = 0 \tag{4.2}$$

and

$$b^{(2)}_{(m,n)} W^{(b,1)}_{(n,2)}(R_1) + \beta^{(2)}_{(m,n)} W^{(b,3)}_{(n,2)}(R_1) = 0 \tag{4.3}$$

where the Z function appearing in (4.2) is defined by (2.16) and (1.30), and the W function appearing in (4.3) is defined by (1.32), (1.33), (2.17), and (3.5). The Z and W functions associated with layer 2 are evaluated at the boundary separating the core from the first coating.

If as before we define $\overline{\overline{Q}}^{(p)}_{(m,n)}$ by the rule

$$\overline{\overline{Q}}^{(p)}_{(m,n)}(R_p) = \overline{\overline{S}}^{(p)-1}_{(m,n)}(R_p)\,\overline{\overline{S}}^{(p+1)}_{(m,n)}(R_p) \tag{4.4}$$

where

$$\overline{\overline{S}}^{(p)}(R_p) \begin{pmatrix} a^{(p)}_{(m,n)} \\ \alpha^{(p)}_{(m,n)} \end{pmatrix} = \begin{pmatrix} Z^{(a,1)}_{(n,p)}(R_p) & Z^{(a,3)}_{(n,p)}(R_p) \\ W^{(a,1)}_{(n,p)}(R_p) & W^{(a,3)}_{(n,p)}(R_p) \end{pmatrix} \begin{pmatrix} a^{(p)}_{(m,n)} \\ \alpha^{(p)}_{(m,n)} \end{pmatrix} \tag{4.5}$$

and

$$\overline{\overline{S}}^{(p+1)}(R_p) \begin{pmatrix} a^{(p+1)}_{(m,n)} \\ \alpha^{(p+1)}_{(m,n)} \end{pmatrix} = \begin{pmatrix} Z^{(a,1)}_{(n,p+1)}(R_p) & Z^{(a,3)}_{(n,p+1)}(R_p) \\ \rho_p W^{(a,1)}_{(n,p+1)}(R_p) & \rho_p W^{(a,3)}_{(n,p+1)}(R_p) \end{pmatrix} \begin{pmatrix} a^{(p+1)}_{(m,n)} \\ \alpha^{(p+1)}_{(m,n)} \end{pmatrix} \tag{4.6}$$

We use these matrices to form the product matrix

$$\overline{\overline{S}}^{(m,n)} = \overline{\overline{Q}}^{(2)}_{(m,n)}(R_2)\,\overline{\overline{Q}}^{(3)}_{(m,n)}(R_3)\ldots\overline{\overline{Q}}^{(N)}_{(m,n)}(R_N) \tag{4.7}$$

Thus the coefficients in layer 2 are related to the expansion coefficients in the outer layer by means of the relation

$$\begin{pmatrix} a^{(2)}_{(m,n)} \\ \alpha^{(2)}_{(m,n)} \end{pmatrix} = \overline{\overline{S}}^{(m,n)} \begin{pmatrix} a^{(N+1)}_{(m,n)} \\ \alpha^{(N+1)}_{(m,n)} \end{pmatrix} \tag{4.8}$$

Thus, using (4.2) and (4.8) we see that for each pair (m,n) of integers where n is nonnegative and the absolute value of m does not exceed n, we have three equations in the three *a priori* unknown coefficients, where only the expansion coefficients, $a^{(N+1)}_{(m,n)}$, are assumed to be known. Thus, if the determinant of the

$\overline{\overline{S}}$ matrix is defined by the rule

$$\Delta\left(\overline{\overline{S}}^{(m,n)}\right) = S_{(2,2)}^{(m,n)} \cdot S_{(1,1)}^{(m,n)} - S_{(1,2)}^{(m,n)} \cdot S_{(2,1)}^{(m,n)} \tag{4.9}$$

then the solution of the system of equations is given by

$$a_{(m,n)}^{(2)} = \left[\frac{\Delta\left(\overline{\overline{S}}^{(m,n)}\right) Z_{(n,2)}^{(a,3)}(R_1)}{S_{(1,2)}^{(m,n)} Z_{(n,2)}^{(a,1)}(R_1) + S_{(2,2)}^{(m,n)} Z_{(n,2)}^{(a,3)}(R_1)}\right] a_{(m,n)}^{(N+1)} \tag{4.10}$$

$$\alpha_{(m,n)}^{(2)} = -\left[\frac{\Delta\left(\overline{\overline{S}}^{(m,n)}\right) Z_{(n,2)}^{(a,1)}(R_1)}{S_{(2,2)}^{(m,n)} Z_{(n,2)}^{(a,1)}(R_1) + S_{(2,2)}^{(m,n)} Z_{(n,2)}^{(a,3)}(R_1)}\right] a_{(m,n)}^{(N+1)} \tag{4.11}$$

and the expansion coefficient of the scattered wave in the medium surrounding the sphere is given by

$$\alpha_{(m,n)}^{(N+1)} = -\left[\frac{S_{(1,1)}^{(m,n)} Z_{(n,2)}^{(a,1)}(R_1) + S_{(2,1)}^{(m,n)} Z_{(n,2)}^{(a,3)}(R_1)}{S_{(1,2)}^{(m,n)} Z_{(n,2)}^{(a,1)}(R_1) + S_{(2,2)}^{(m,n)} Z_{(n,2)}^{(a,3)}(R_1)}\right] a_{(m,n)}^{(N+1)} \tag{4.12}$$

The remaining expansion coefficients are derived in an analogous manner. By defining a matrix $\overline{\overline{T}}^{(m,n)}$ by the rule

$$\overline{\overline{T}}^{(m,n)} = \overline{\overline{R}}_{(m,n)}^{(2)}(R_2)\, \overline{\overline{R}}_{(m,n)}^{(3)}(R_3) \ldots \overline{\overline{R}}_{(m,n)}^{(N)}(R_N) \tag{4.13}$$

where the $\overline{\overline{R}}$ matrices appearing in (4.13) are defined by (3.9), (3.10), and (3.12), we see that we have the relationship between the expansion coefficients in the first penetrable layer surrounding the perfectly conducting core and the expansion coefficients in the region of free space surrounding the multilayer spherical structure given by

$$\begin{pmatrix} b_{(m,n)}^{(2)} \\ \beta_{(m,n)}^{(2)} \end{pmatrix} = \overline{\overline{T}}^{(m,n)} \begin{pmatrix} b_{(m,n)}^{(N+1)} \\ \beta_{(m,n)}^{(N+1)} \end{pmatrix} \tag{4.14}$$

For every pair (m, n), the equation (4.14) and the relationship (4.3) demanded by the assertion that the tangential component of $\overline{E}$ vanish on the boundary of the scatterer gives us, as before, three equations in three unknowns. The solution of this system of equations is

$$b_{(m,n)}^{(2)} = \left[\frac{\Delta\left(\overline{\overline{T}}^{(m,n)}\right) W_{(n,2)}^{(b,3)}(R_1)}{T_{(1,2)}^{(m,n)} W_{(n,2)}^{(b,1)}(R_1) + T_{(2,2)}^{(m,n)} W_{(n,2)}^{(b,3)}(R_1)}\right] b_{(m,n)}^{(N+1)} \tag{4.15}$$

$$\beta^{(2)}_{(m,n)} = -\left[\frac{\Delta\left(\overline{\overline{T}}^{(m,n)}\right) W^{(b,1)}_{(n,2)}(R_1)}{T^{(m,n)}_{(1,2)} W^{(b,1)}_{(n,2)}(R_1) + T^{(m,n)}_{(2,2)} W^{(b,3)}_{(n,2)}(R_1)}\right] b^{(N+1)}_{(m,n)} \tag{4.16}$$

and the expansion coefficient of the scattered wave in the medium surrounding the sphere is given by

$$\beta^{(N+1)}_{(m,n)} = -\left[\frac{T^{(m,n)}_{(1,1)} W^{(b,1)}_{(n,2)}(R_1) + T^{(m,n)}_{(2,1)} W^{(b,3)}_{(n,2)}(R_1)}{T^{(m,n)}_{(1,2)} W^{(b,1)}_{(n,2)}(R_1) + T^{(m,n)}_{(2,2)} W^{(b,3)}_{(n,2)}(R_1)}\right] b^{(N+1)}_{(m,n)} \tag{4.17}$$

where the determinant $\Delta\left(\overline{\overline{T}}^{(m,n)}\right)$ is given by

$$\Delta\left(\overline{\overline{T}}^{(m,n)}\right) = T^{(m,n)}_{(2,2)} \cdot T^{(m,n)}_{(1,1)} - T^{(m,n)}_{(1,2)} \cdot T^{(m,n)}_{(2,1)} \tag{4.18}$$

This analysis enables us to assess the effectiveness of coatings on materials in impeding or transmitting radiation to a conducting core surrounded by penetrable but anisotropic materials with both nontrivial electrical and nontrivial magnetic properties. We can also calculate the absorbed and scattered radiation by using the expansion coefficients for the scattered radiation in the outer layer. The electric vector of the incident radiation is expressed in terms of known expansion coefficients by the relation

$$\begin{aligned}\overline{E}^i = \sum_{(m,n)\in I} \Bigg\{ & a^{(N+1)}_{(m,n)} Z^{(0,1)}_{(n,N+1)}(r)\, \overline{A}_{(m,n)}(\theta,\phi) \\ & + b^{(N+1)}_{(m,n)} \left(-\frac{1}{k_0 r}\right) \frac{\partial}{\partial r}\left(r Z^{(0,1)}_{(n,N+1)}(r)\right) \overline{B}_{(m,n)}(\theta,\phi) \\ & - n(n+1)\, b^{(N+1)}_{(m,n)} \frac{Z^{(0,1)}_{(n,N+1)}(r)}{k_0 r} \overline{C}_{(m,n)}(\theta,\phi) \Bigg\}\end{aligned} \tag{4.19}$$

where the superscript zero for the Z functions and their derivatives means that these functions are solutions of the ordinary Bessel equation (2.27) with $k = k_0$ All other expansion coefficients are expressed in terms of the expansion coefficients appearing in (4.19).

5. LAYERS SEPARATED BY IMPEDANCE SHEETS OR CHARGE LAYERS

The impedance sheet relationship states that at the surface

$$r = R_p \tag{5.1}$$

separating region p from region $p+1$ we have

$$\hat{e}_r \times (\overline{H}_{p+1} - \overline{H}_p) = \sigma_s^{(p)} \left[\overline{E}_p - (\overline{E}_p \cdot \hat{e}_r)\hat{e}_r\right] \tag{5.2}$$

where the quantity $\sigma_s^{(p)}$ is the surface conductivity of a sheet covering the sphere bounding the pth layer, where the real part is a positive quantity that indicates

the level of the loss associated with moving electrons along this surface, and the imaginary part of $\sigma_s^{(p)}$ denotes an inductive or a reactive effect of the sheet, depending upon its sign.

The development of intralayer relationships through the use of equation (5.2) is enhanced if we simply use the definitions (2.5), (2.6), and (2.7) and observe that

$$\hat{e}_r \times \overline{C}_{(m,n)}(\theta, \phi) = 0 \tag{5.3}$$

$$\hat{e}_r \times \overline{B}_{(m,n)}(\theta, \phi) = -\overline{A}_{(m,n)}(\theta, \phi) \tag{5.4}$$

and

$$\hat{e}_r \times \overline{A}_{(m,n)}(\theta, \phi) = \overline{B}_{(m,n)}(\theta, \phi) \tag{5.5}$$

Making use of the above relationships, (5.3) through (5.5) and (3.2) which gives a representation

$$\begin{aligned}
&\hat{e}_r \times (\overline{H}_{p+1} - \overline{H}_p) \\
&= \sum_{(m,n)\in I} \Bigg\{ (-1) \Bigg[\frac{ik_{p+1}}{\omega\mu^{(p+1)}}\, a_{(m,n)}^{(p+1)}\, W_{(n,p+1)}^{(a,1)}(R_p) - \frac{ik_p}{\omega\mu^{(p)}}\, a_{(m,n)}^{(p)}\, W_{(n,p)}^{(a,1)}(R_p) \\
&+ \frac{ik_{p+1}}{\omega\mu^{(p+1)}}\, \alpha_{(m,n)}^{(p+1)}\, W_{(n,p+1)}^{(a,3)}(R_p) - \frac{ik_p}{\omega\mu^{(p)}}\, \alpha_{(m,n)}^{(p)}\, W_{(n,p)}^{(a,3)}(R_p) \Bigg] \overline{A}_{(m,n)}(\theta, \phi) \\
&+ \Bigg[\frac{ik_{p+1}}{\omega\mu^{(p+1)}}\, b_{(m,n)}^{(p+1)}\, Z_{(n,p+1)}^{(b,1)}(R_p) - \frac{ik_p}{\omega\mu^{(p)}}\, b_{(m,n)}^{(p)}\, Z_{(n,p)}^{(b,1)}(R_p) \\
&+ \frac{ik_{p+1}}{\omega\mu^{(p+1)}}\, \beta_{(m,n)}^{(b+1)}\, Z_{(n,p+1)}^{(b,3)}(R_p) - \frac{ik_p}{\omega\mu^{(p)}}\, \beta_{(m,n)}^{(p)}\, Z_{(n,p)}^{(b,3)}(R_p) \Bigg] \overline{B}_{(m,n)}(\theta, \phi) \Bigg\}
\end{aligned} \tag{5.6}$$

If we assume a tensor relationship between the surface conductivity and the electric vector we obtain the relationship

$$\begin{aligned}
\overline{J}_e = \sum_{(m,n)\in I} \Bigg\{ &\sigma_a \left[a_{(m,n)}^{(p)} Z_{(n,p)}^{(a,1)} + \alpha_{(m,n)}^{(p)} Z_{(n,p)}^{(a,3)} \right] (R_p)\, \overline{A}_{(m,n)}(\theta, \phi) \\
&+ \sigma_b \left[-b_{(m,n)}^{(p)} W_{(n,p)}^{(b,1)} - \beta_{(m,n)}^{(p)} W_{(n,p)}^{(b,3)} \right] (R_p)\, \overline{B}_{(m,n)}(\theta, \phi) \Bigg\}
\end{aligned} \tag{5.7}$$

By making use of orthogonality we obtain the following relationships

$$\begin{aligned}
&\rho_p\, a_{(m,n)}^{(p+1)}\, W_{(n,p+1)}^{(a,1)}(R_p) + \rho_p\, \alpha_{(m,n)}^{(p+1)}\, W_{(n,p+1)}^{(a,3)}(R_p) \\
&= a_{(m,n)}^{(p)}\, W_{(n,p)}^{(a,1)}(R_p) + \alpha_{(m,n)}^{(p)}\, W_{(n,p)}^{(a,3)}(R_p) \\
&\quad + \frac{i\omega\mu^{(p)}\sigma_a}{k_p} \left[a_{(m,n)}^{(p)}\, Z_{(n,p)}^{(a,1)}(R_p) + \alpha_{(m,n)}^{(p)}\, Z_{(n,p)}^{(a,3)}(R_p) \right]
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
&\rho_p\, b_{(m,n)}^{(p+1)}\, Z_{(n,p+1)}^{(b,1)}(R_p) + \rho_p\, \beta_{(m,n)}^{(p+1)}\, Z_{(n,p+1)}^{(b,3)}(R_p) \\
&= b_{(m,n)}^{(p)}\, Z_{(n,p)}^{(b,1)}(R_p) + \beta_{(m,n)}^{(p)}\, Z_{(n,p)}^{(b,3)}(R_p)
\end{aligned}$$

$$+(-1)\frac{i\omega\mu^{(p)}\sigma_b}{k_p}\left[b^{(p)}_{(m,n)}W^{(b,1)}_{(n,p)}(R_p)+\beta^{(p)}_{(m,n)}W^{(b,3)}_{(n,p)}(R_p)\right] \quad (5.9)$$

In order to simplify the matrix relationships connecting expansion coefficients in layer p to those in layer $p+1$ we introduce some new functions by the rules

$$U^{(a,j)}_{(n,p)}=\left[W^{(a,j)}_{(n,p)}+\left(\frac{i\omega\mu^{(p)}\sigma_a}{k_p}\right)Z^{(a,j)}_{(n,p)}\right](R_p) \quad (5.10)$$

which arises naturally from (5.8) and

$$V^{(b,j)}_{(n,p)}=\left[Z^{(b,j)}_{(n,p)}+\left(\frac{-i\omega\mu^{(p)}\sigma_b}{k_p}\right)W^{(b,j)}_{(n,p)}\right](R_p) \quad (5.11)$$

which is based on (5.9).

Making use of the definitions in (5.10) and (5.11) we reproduce a setting similar to either that of Section 3 or Section 4 depending on whether or not the inner core is penetrable or perfectly conducting, respectively. The matrix relationships are

$$\begin{pmatrix} Z^{(a,1)}_{(n,p)}(R_p) & Z^{(a,3)}_{(n,p)}(R_p) \\ U^{(a,1)}_{(n,p)}(R_p) & U^{(a,3)}_{(n,p)}(R_p)\end{pmatrix}\begin{pmatrix} a^{(p)}_{(m,n)} \\ \alpha^{(p)}_{(m,n)}\end{pmatrix}=$$

$$\begin{pmatrix} Z^{(a,1)}_{(n,p+1)}(R_p) & Z^{(a,3)}_{(n,p+1)}(R_p) \\ \rho_p W^{(a,1)}_{(n,p+1)}(R_p) & \rho_p W^{(a,3)}_{(n,p+1)}(R_p)\end{pmatrix}\begin{pmatrix} a^{(p+1)}_{(m,n)} \\ \alpha^{(p+1)}_{(m,n)}\end{pmatrix} \quad (5.12)$$

and

$$\begin{pmatrix} V^{(b,1)}_{(n,p)}(R_p) & V^{(b,3)}_{(n,p)}(R_p) \\ W^{(b,1)}_{(n,p)}(R_p) & W^{(b,3)}_{(n,p)}(R_p)\end{pmatrix}\begin{pmatrix} b^{(p)}_{(m,n)} \\ \beta^{(p)}_{(m,n)}\end{pmatrix}=$$

$$\begin{pmatrix} \rho_p Z^{(b,1)}_{(n,p+1)}(R_p) & \rho_p Z^{(b,3)}_{(n,p+1)}(R_p) \\ W^{(b,1)}_{(n,p+1)}(R_p) & W^{(b,3)}_{(n,p+1)}(R_p)\end{pmatrix}\begin{pmatrix} b^{(p+1)}_{(m,n)} \\ \beta^{(p+1)}_{(m,n)}\end{pmatrix} \quad (5.13)$$

We note that in case

$$0=\sigma_a=\sigma_b \quad (5.14)$$

the intralayer relationships (5.12) and (5.13) are exactly those given by (3.7) and (3.9), respectively. The matrix equations are based on the impedance sheet relation, (5.2), which is decomposed through orthogonality to the relations based on (5.8) through (5.11). The other relation giving rise to (5.12) and (5.13) is the continuity of tangential components of $\vec{E}$ across the surface separating regions of continuity of electrical properties, namely, those defined by

$$r=R_p \quad (5.15)$$

The continuity of tangential components of $\overline{E}$ across the surface defined by (5.15) may be expressed as

$$\hat{e}_r \times \overline{E}_p = \hat{e}_r \times \overline{E}_{p+1} \tag{5.16}$$

Equation (5.16) which holds at the separating surface defined by (5.15) is reduced through the use of orthogonality to the relations

$$\begin{aligned} &a^{(p)}_{(m,n)} Z^{(a,1)}_{(n,p)}(R_p) + \alpha^{(p)}_{(m,n)} Z^{(a,3)}_{(n,p)}(R_p) \\ &\quad = a^{(p+1)}_{(m,n)} Z^{(a,1)}_{(n,p+1)}(R_p) + \alpha^{(p+1)}_{(m,n)} Z^{(a,3)}_{(n,p+1)}(R_p) \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} &b^{(p)}_{(m,n)} W^{(b,1)}_{(n,p)}(R_p) + \beta^{(p)}_{(m,n)} W^{(b,3)}_{(n,p)}(R_p) \\ &\quad = b^{(p+1)}_{(m,n)} W^{(b,1)}_{(n,p+1)}(R_p) + \beta^{(p+1)}_{(m,n)} W^{(b,3)}_{(n,p+1)}(R_p) \end{aligned} \tag{5.18}$$

In the computer program we check the boundary conditions after the system of equations involving the expansion coefficients is solved. Another check that is made is the energy balance relationship. This is described in the next section.

The solution of the system of equations relating the expansion coefficients is therefore the same as the solutions given in Section 3 and Section 4 and that the Wronskian relationship shows that the determinant of the new coefficient matrices on the left sides of (5.12) and (5.13) are respectively equal to the determinant of the coefficient matrices of the left sides of (3.7) and (3.8). The inverses needed to define the analogue of the matrices defined by (3.11) and (3.12) have to exist.

6. ENERGY BALANCE RELATIONSHIP

In this section of our paper we demonstrate a powerful means of validating computer codes for describing the interaction of electromagnetic radiation with anisotropic structures whose regions of continuity of tensorial electromagnetic properties are delimited by spherical reactive and lossy impedance sheets with a common center. This was important since no previous workers have published any data regarding scattering by anisotropic spheres. We have found data concerning scattering by isotropic spheres with either a penetrable or a perfectly conducting core, but the standard references do not include calculations of bistatic cross sections of magnetically lossy spherical structures even when the layers are isotropic, and no energy balance computations for these magnetically lossy structures seems to have been published. This section provides data for magnetically lossy anisotropic structures. We keep track of the energy going into our structure and the radiation scattered away from our spherically symmetric multilayer structure. This can be carried out by a Poynting vector analysis over the outermost surface of the scatterer or by integrating the power density over the interior of the layers or the surfaces of lossy impedance sheets. We discuss here a simple calculation when impedance sheets are present. The sphere considered, also may have both magnetic and electrical losses. The constitutive relations are tensorial. In what follows if α is a tensor, then $Re(\alpha)$ is the tensor formed by computing the real part of each entry and $Im(\alpha)$ is the tensor formed by computing the

imaginary part of each entry. When there are off-diagonal elements it is possible to have losses, for example, from a permittivity tensor which has only real entries. If σ, ϵ, and μ, are respectively the conductivity, permittivity, and permeability tensors, then losses arise, for diagonal tensors, from the real part of the conductivity tensor, the imaginary part of the permittivity tensor, and the imaginary part, $\mathrm{Im}(\mu)$, of the permeability tensor.

The sum of the surface integrals of the energy per unit area per unit time deposited in the impedance sheets separating the regions of continuity of the tensorial electromagnetic properties of the material plus the triple integral of the power density over the interior of an anisotropic multilayer spherical structure is related to the expansion coefficients $\alpha_{(n,N+1)}$ and $\beta_{(n,N+1)}$ of the scattered radiation by the equation [1],

$$\begin{aligned}
&\sum_{k=1}^{M}\left\{R_k^2\int_0^{\pi}\int_0^{2\pi}\left[\mathrm{Re}(\sigma_s^{(k)})\left(|E_\theta|^2+|E_\phi|^2\right)(R_k,\theta,\phi)\right]d\phi\sin\theta\,d\theta\right\}\\
&\quad+\int_0^R\int_0^{\pi}\int_0^{2\pi}\left[\omega\,\mathrm{Im}(\mu)\left(|H_\theta|^2+|H_\phi|^2\right)+\omega\,\mathrm{Im}(\mu_r)|H_r|^2\right]r^2\sin\theta\,d\phi\,d\theta\,dr\\
&\quad+\int_0^R\int_0^{\pi}\int_0^{2\pi}\left[\omega\mathrm{Im}(\epsilon)\left(|E_\theta|^2+|E_\phi|^2\right)+\omega\,\mathrm{Im}(\epsilon_r)|E_r|^2\right]r^2\sin\theta\,d\phi\,d\theta\,dr\\
&\quad+\int_0^R\int_0^{\pi}\int_0^{2\pi}\left[\mathrm{Re}(\sigma)\left(|E_\theta|^2+|E_\phi|^2\right)+\mathrm{Re}(\sigma_r)|E_r|^2\right]r^2\sin\theta\,d\phi\,d\theta\,dr\\
&=\frac{\pi|E_0^2|}{k_0^2}\left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}}\left|\mathrm{Re}\sum_{n=1}^{\infty}(2n+1)\left(\alpha_{(n,N+1)}+\beta_{(n,N+1)}\right)\right|\\
&\quad-\frac{\pi|E_0^2|}{k_0^2}\left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}}\sum_{n=1}^{\infty}(2n+1)\left(|\alpha_{(n,N+1)}|^2+|\beta_{(n,N+1)}|^2\right)
\end{aligned}\tag{6.1}$$

The constitutive relations between $\partial\overline{D}/\partial t+\overline{J}$ and $\overline{E}$, for the case of time harmonic radiation are given by (1.22), and the constitutive relations between $\partial\overline{B}/\partial t$ and the magnetic field $\overline{H}$ are defined by (1.23).

The above computations have enabled us to provide a reliable set of equations for the expansion coefficients of both the scattered and induced electromagnetic fields. If there were an error in the formulas that produced incorrect electromagnetic fields, one could be confident that integrating an incorrect power density over one of the layers or over one of the delimiting impedance sheets would not have given a 7 to 10 digit agreement between the computations of the total absorbed power using an energy balance computation by numerically integrating the power density distribution over the layers and over the surfaces of the lossy impedance sheets and making a comparison between the Poynting vector calculation on the outer surface of the sphere of the total electromagnetic energy minus the total energy scattered away from the spherical structure. These computations should enable one to quickly identify any errors in algebra or transcription of the formulas to the computer algorithm.

One of the unique features of our anisotropic structure was the peculiar behav-

ior of the fields near the center of the sphere. It turns out that if we have only an anisotropy in the real part of the permittivity and if the radial component exceeds the angular component, then there is an integrable singularity in the power density at the center and if the radial permittivity is smaller than the angular permittivity, then the electric field vector goes to zero at the origin. Furthermore, if the radial permittivity is larger than the angular permittivity, no combination of the two linearly independent solutions will yield a solution which is bounded at the origin.

In spite of the complexities of the interaction of radiation with anisotropic structures we found that we could achieve the 9 digit agreement in the two methods of computing the total absorbed power with a tensor product of Gaussian quadrature schemes and for the most part making use of only 1728 function evaluations per layer. We checked the Gaussian quadrature by using in addition scheme which used 4096 function evaluations per layer. We checked the Gaussian quadrature by using in addition a scheme which used 4096 function evaluations per layer and make sure that the two methods gave the same answer. We also checked the computations by artificially adding extra layers. In making the computations we also made certain that the exact formula for the incoming radiation and our series expansion in spherical harmonics gave numerically identical answers on the surface of the sphere, and had checking subroutines which calculated the tangential components of the electric and magnetic vectors on both sides of the delimiting surface. When the delimiting surface happened to be an impedance sheet we not only made sure that the impedance sheet boundary condition was satisfied, we also checked to make sure that the spherical harmonic expansion of the internal fields had converged.

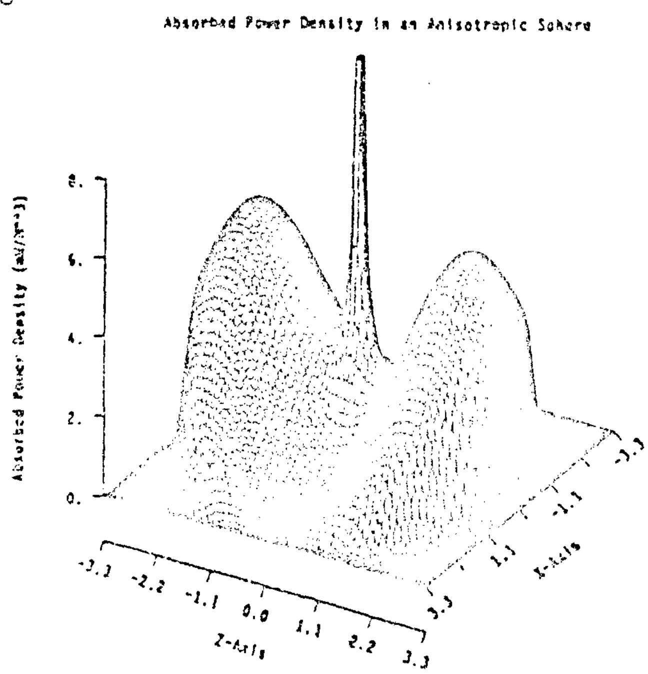

Figure 1(a).

APPENDIX

The absorption of electromagnetic radiation by anisotropic spherical structures is strikingly different from that of isotropic structures. To illustrate this we consider the computer calculations carried out for a single layer nonmagnetic spherical structure with a relative radial permittivity of 40 and a relative tangential permittivity of 20 and a slight loss of .005 mhos per meter subjected to 1 Gigahertz radiation and with a radius of 3 centimeters; the structure was exposed to plane wave radiation with a strength of 1 volt per meter, and the radiation was traveling in the direction of the positive z-axis. Figure 1 shows an absorption peak in the center of the spherical structure; the three dimensional plot was smoothed by allowing the power density to gradually go to zero on the edge of a circle with a radius of 3.3 centimeters using linear interpolation from the value at 3 centimeters. The values in the interior of the 3 centimeter sphere are exact. Figure 2 shows the same power density distribution when the relative radial permittivity is 20 and the relative tangential permittivity is 40; in this case there is a null in the power density distribution in the center of the spherical structure; the anisotropy acts as a type of energy shield for the point at the center of this structure, while in the situation depicted in Fig. 1, it serves to focus the electromagnetic energy. These computations were validated by the energy balance computations suggested by equation (6.1). In this equation the R shown in the integral is actually R_N, the radius of the Nth and last spherical shell, the radial permittivity is ϵ_r and the tangential permittivity is ϵ; the tangential permeability is μ, and the radial permeability is μ_r. We also make use of a complex conductivity whose radial and tangential values are σ_r and σ, respectively. The impedance sheet conductivity

ELECTROMAGNETIC PARAMETER	REAL PART	IMAGINARY PART
Radial permittivity (relative)	10.00	5.00
Tangential permittivity (relative)	15.00	7.00
Radial permeability (relative)	21.00	2.00
Tangential permeability (relative)	11.00	3.00
Radial conductivity (mhos per meter)	3.00	1.00
Tangential conductivity (mhos per meter)	2.00	2.00
Impedance sheet conductivity and reactivity at a radius of 3 centimeters	1.00	3.20

TOTAL ABSORBED POWER

BY VOLUME AND SURFACE INTEGRATION: THE SUM OF THE DIELECTRIC, MAGNETIC, CONDUCTIVE, AND IMPEDANCE SHEET LOSSES	BY A POYNTING VECTOR ANALYSIS; THE ENTERING POWER MINUS THE SCATTERED POWER
$6.95697953 \times 10^{-9}$ Watts	$6.95697953 \times 10^{-9}$ Watts

Table 1. An energy balance analysis for an anisotropic spherical scatterer with a radius of 3 centimeters subjected to 1 Gigahertz radiation with a field strength of 1 volt per meter.

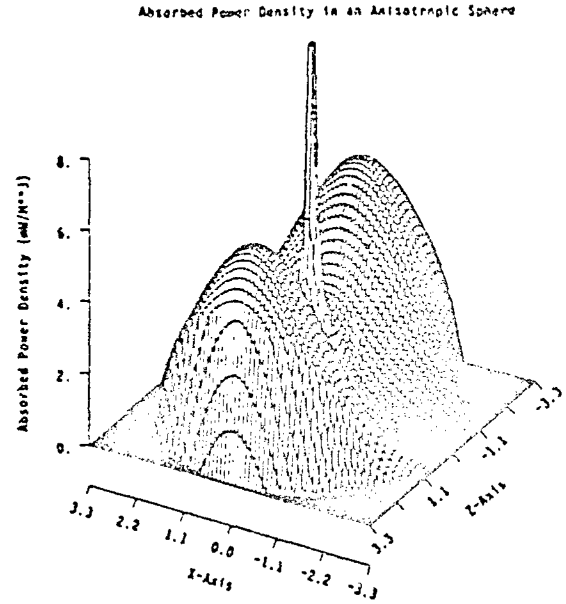

Figure 1(b).

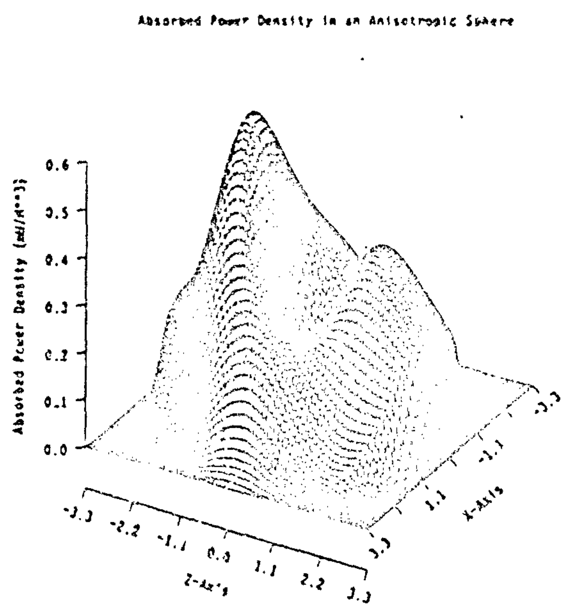

Figure 2.

is defined on the kth spherical surface by a single complex number $\sigma_s^{(k)}$. The computer code was validated using Equation (6.1) by making runs with balloons formed by impedance sheets; we considered both reactive and lossy impedance sheets and situations where there were concentric spherical impedance sheets sep-

arated by regions of free space. An area of future research includes the investigation of radiative heating of anisotropic spherical structures when the thermal conductivity is different in radial and tangential directions. Another application of the code is the investigation of the range of validity of impedance boundary conditions by making plots of the ratio of the norm of the tangential component of the electric vector to the norm of the tangential component of the magnetic vector along curves on the outer surface of the sphere.

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D. K. Cohoon was a graduate of Massachusetts Institute of Technology. He served at Argonne National Laboratory, was stationed at Ft. Meade, and served at Bell Telephone Laboratories, the University of Wisconsin Department of Mathematics, the University of Minnesota Department of Mathematics, the School of Aerospace Medicine in San Antonio, and Temple University.

Determination of the Effect of Transient, Spatially Heterogeneous Electromagnetic Radiation on a Realistic Model of Man

D. K. Cohoon

March 1, 1992

We consider in this paper powerful general methods for solving electromagnetic scattering problems that would with presently used methods require 1000 human life times of time on an advanced computer. After implementation of the advanced methods proposed in this paper only one hour would be required on the same computer.

We consider first demonstrate the capability of modeling the interaction of radiation with a bounded three dimensional body, covered with impedance sheets and having full tensor bianisotropy (e.g. a moving astronaut wearing a protective device) using a volume integral equation approach that will permit modeling of tissue heterogeneities. The EFRJE, exact finite rank integral equation, approach is used which will permit machine precision results once a reasonably close approximation is obtained.

As a specific and easily understood example, (to illustrate the new EFRIE method) we consider the discretization of the integral equation of electromagnetic scattering for a magnetic, but penetrable structure delimited by parallel planes.

The volume integral equation approach would still tax even the most advanced computers if a classical method of moments (weak topology convergence) approach were used, and neither the sponsor or the performer of the research would know the accuracy of the internal fields, and consequently we propose the use of a surface integral equation approach by using a combined field integral equation approach to calculate electrical and magnetic

currents and electrical and magnetic charges on the surfaces bounding the exterior of the human body and the internal organs, the heart, lungs, liver, spleen, kidneys and their cavities under the assumption that these individual organs have homogeneous permittivity, conductivity, and permeability. Since we are carrying out a discretization of a surface rather than a volume this should greatly reduce the computational complexity and give us the potential, when implemented with the EFRIE approach, for obtaining highly accurate results. We could model a complex of tissue components or open spaces with the property that each region of homogeneity of tissue components may be as general as a diffeomorphm of the interior of an N handled sphere.

Contents

1 INTRODUCTION

We shall in this paper consider powerful new methods for formulating and solving integral equations describing the interaction of electromagnetic radiation with complex materials. Such interaction problems, for currently used methods, such as the method of moments, are beyond the capability of existing computers.

1.1 Integral Equations and Bianisotropy

Bianisotropic materials, because of their greater complexity, have greater potential for creating materials with prescribed or desired absorption, transmission, and reflection properties. Chiral properties are a special case of bianisotropic materials. With chiral materials there is a special scalar ξ_c (Jaggard and Engheta, p 173) such that

$$\boldsymbol{D} = \epsilon \boldsymbol{E} + i\xi_c \boldsymbol{B} \tag{1.1.1}$$

and

$$\boldsymbol{B} = \mu \boldsymbol{H} - i\xi_c \mu \boldsymbol{E} \tag{1.1.2}$$

With the more general bianisotropic materials there are tensors α and β with the property that

$$\boldsymbol{D} = \epsilon \boldsymbol{E} + \alpha \boldsymbol{H}/(i\omega) \tag{1.1.3}$$

and

$$\boldsymbol{B} = \mu \boldsymbol{H} + \beta \boldsymbol{E}/(i\omega) \tag{1.1.4}$$

where ϵ and μ are tensors. Here Maxwell's equations have the form

$$curl(E) = -i\omega B \tag{1.1.5}$$

and

$$curl(H) = i\omega D + \sigma E \tag{1.1.6}$$

Using these notions we make Maxwell's equations look like the standard Maxwell equations with complex sources by introducing the generalized electric and magnetic current densities by the relations,

$$curl(E) = i\omega\mu_0 H - J_m \tag{1.1.7}$$

and

$$curl(H) = i\omega\epsilon_0 E + J_e \tag{1.1.8}$$

where

$$J_e = i\omega\epsilon E + \alpha H - i\omega\epsilon_0 E \tag{1.1.9}$$

and

$$J_m = i\omega\mu H + \beta E - i\omega\mu_0 H \tag{1.1.10}$$

The formulation of integral equations for bianisotropic materials, therefore, is carried out by the analysis of the following coupled system of integral equations based on the notion of electric and magnetic charges defined by the two continuity equations

$$div(J_e) + \frac{\partial \rho_e}{\partial t} \tag{1.1.11}$$

and

$$div(J_m) + \frac{\partial \rho_m}{\partial t} \tag{1.1.12}$$

Having developed this the coupled system of integral equations describing the interaction of electromagnetic radiation with a bounded bianisotropic body Ω is given by the following relations. The electric field integral equation is given by

$$\begin{aligned} E - E^i &= - grad\left(\int_\Omega \frac{div(J_e)}{\omega\epsilon_0} G(r,s)dv(s)\right) \\ &+ \frac{i}{\omega\epsilon_0} grad\left(\int_{\partial\Omega} (J_e \cdot n) G(r,s)da(s)\right) \end{aligned}$$

$$
\begin{aligned}
&- i\omega\mu_0 \int_\Omega \boldsymbol{J}_e G(r,s) dv(s) \\
&- curl \left(\int_\Omega \boldsymbol{J}_m G(r,s) dv(s) \right)
\end{aligned} \tag{1.1.13}
$$

and the magnetic field integral equation may be expressed as

$$
\begin{aligned}
\boldsymbol{H} - \boldsymbol{H}^i = &-grad \left(\int_\Omega \frac{div(\boldsymbol{J}_m)}{\omega\mu_0} G(r,s) dv(s) \right) \\
&-\frac{i}{\omega\mu_0} grad \left(\int_{\partial\Omega} (\boldsymbol{J}_m \cdot n) G(r,s) da(s) \right) \\
&-i\omega\epsilon_0 \int_\Omega \boldsymbol{J}_m G(r,s) dv(s) + \\
&curl \left(\int_\Omega \boldsymbol{J}_e G(r,s) dv(s) \right)
\end{aligned} \tag{1.1.14}
$$

where G(r,s) is the rotation invariant, temperate fundamental solution of the Helmholtz equation,

$$
(\Delta + k_0^2) G = \delta \tag{1.1.15}
$$

given by

$$
G(r,s) = \frac{exp(-ik_0 \mid r - s \mid)}{4\pi \mid r - s \mid} \tag{1.1.16}
$$

Substituting (1.1.9) and (1.1.10) into equations (1.1.13) and (1.1.14) we obtain, the coupled integral equations for bianisotropic materials. The electric field integral equation for a bianisotropic material is given by,

$$
\begin{aligned}
&\boldsymbol{E} - \boldsymbol{E}^i = \\
&-grad \left(\int_\Omega \frac{div(i\omega\epsilon \boldsymbol{E} + \alpha \boldsymbol{H} - i\omega\epsilon_0 \boldsymbol{E}}{\omega\epsilon_0} G(r,s) dv(s) \right) \\
&+ \frac{i}{\omega\epsilon_0} grad \left(\int_{\partial\Omega} (i\omega\epsilon \boldsymbol{E} + \alpha \boldsymbol{H} - i\omega\epsilon_0 \boldsymbol{E} \cdot n) G(r,s) da(s) \right) \\
&- i\omega\mu_0 \int_\Omega i\omega\epsilon \boldsymbol{E} + \alpha \boldsymbol{H} - i\omega\epsilon_0 \boldsymbol{E} G(r,s) dv(s) + \\
&- curl \left(\int_\Omega i\omega\mu \boldsymbol{H} + \beta \boldsymbol{E} - i\omega\mu_0 \boldsymbol{H} G(r,s) dv(s) \right)
\end{aligned} \tag{1.1.17}
$$

and the magnetic field integral equation for a bianisotropic material is given by

$$
\boldsymbol{H} - \boldsymbol{H}^i = -grad \left(\int_\Omega \frac{div(i\omega\mu \boldsymbol{H} + \beta \boldsymbol{E} - i\omega\mu_0 \boldsymbol{H}}{\omega\mu_0} G(r,s) dv(s) \right)
$$

$$
\begin{aligned}
&- \frac{i}{\omega\mu_0} grad \int_{\partial\Omega} (i\omega\mu H + \beta E - i\omega\mu_0 H \cdot n) G(r,s) da(s) \\
&- i\omega\epsilon_0 \int_\Omega (i\omega\mu H + \beta E - i\omega\mu_0 H) G(r,s) dv(s) + \\
&curl \left(\int_\Omega i\omega\epsilon E + \alpha H - i\omega\epsilon_0 E G(r,s) dv(s) \right) \qquad (1.1.18)
\end{aligned}
$$

In the subsequent sections we shall explore methods of resolving these integral equations on existing computers using novel, powerful analytical methods of solution.

2 EFRIE Methods

While we have obtained exact solutions for layered materials, most of the problems are so complex that one must formulate the interaction problems using integral equations. The primary focus of this report is to describe the design of complex materials using an improvement of classical spline methods (Tsai, Massoudi, Durney, and Iskander, pp 1131-1139). This paper is unusual in that comparisons are made between internal fields predicted from moment method computations and Mie solution computations. Successful comparisons have been made for linear basis functions without enhancement by EFRIE theory. However, as the scattering bodies become more complex the computational requirements become larger and larger. With EFRIE theory if one has a discretization that enables one to closely approximate the solution, then refinements can be made by a convergent iterative process based on the concept that the norm of the difference between an approximate integral operator and the actual integral operator is simply smaller than one, not necessarily close enough to give answers of acceptable accuracy. Then the answer is improved by an iterative process to any desired precision without the use of additional computer memory.

2.1 Examples of Spaces of Approximation

Solving the electromagnetic transmission problem by finding solutions of Maxwell's equations inside and outside a penetrable scatterer which satisfy boundary conditions and

radiation conditions requires functions on a continuum, the problem is from a practical point of view a discrete one and involves estimation of the values of induced and scattered electric and magnetic vectors in the interior and the exterior of the scattering body. Thus, it is important to understand methods of determining the accuracy with which a solution of a discrete approximation of an integral equation formulation of an electromagnetic interaction problem can be obtained. We specifically need to formulate a space of approximates and a projection operator onto this space of approximates and formulate a finite rank approximation of the original infinite rank integral equations (1.1.17) and (1.1.18) such that the precise solution of this approximate equation is exactly the projection onto the space of approximates of the solution of the original infinite rank integral equation. We further need to develop a means of correcting our solution so that we may exactly determine by iteration the difference $f - Pf$ between the solution f of the original equation and the projection Pf of this solution onto the space of approximates, possibly by an iterative scheme or a series expansion. In this section we illustrate (i) pulse basis function methods, (ii) linear interpolation, (iii) higher order spline interpolation, and (iv) a completetely novel L^∞ norm method of approximating the field components with combinations of trignomometric functions of the local spatial variables using carefully selected frequencies.

We now explain linear interpolation. A common example would be to approximate the space V of functions which are continuous on $[a, b]$ by members of a set

$$S = \{[x_0, x_1), [x_1, x_2), \cdots, [x_{n-1}, x_n]\} \tag{2.1.1}$$

where

$$a = x_0 < x_1 < \cdot cdot \cdot < x_n = b \tag{2.1.2}$$

and to define the projection operator of linear interpolation, for the partition defined by equation (2.1.1) by the rule,

$$Pf(x) = f(x_{i-1})\left(\frac{x_i - x}{x_i - x_{i-1}}\right) + f(x_i)\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) \tag{2.1.3}$$

if x belongs to the subinterval from x_{i-1} to x_i and we note that if this is the case then since

$$(Pf)(x_{i-1}) = f(x_{i-1}) \cdot 1 + 0 \tag{2.1.4}$$

and since

$$(Pf)(x_i) = 0 + f(x_i) \cdot 1 \tag{2.1.5}$$

it follows from equations (2.1.3), (2.1.4), and (2.1.5) that

$$P^2 f = Pf \tag{2.1.6}$$

Another simple example is Fourier series or an eigenfunction expansion in the spatial variables. Suppose that V is a set of functions defined on $\mathbb{R}^n$ which are square integrable with respect to Lebesgue measure ν multiplied by a positive function ρ and valued in a Hilbert space X with norm $| \cdot |_X$ with two measurable and square integrable functions f and g being equivalent on an open set,

$$\Omega \subset \mathbb{R}^n, \tag{2.1.7}$$

if and only if

$$\int_\Omega \left(| (f-g)(x) |_X^2 \right) \rho(x) d\nu(x) = 0 \tag{2.1.8}$$

and where the square integrability with respect to the ordinary Lebesgue measure multiplied by ρ means that

$$\int_\Omega \left(| f(x) |_X^2 \right) \rho(x) d\nu(x) < \infty \tag{2.1.9}$$

We say that two Hilbert space valued functions f and g are orthogonal if and only if

$$\int_\Omega \{ f(x) \cdot g(x) \} \rho(x) d\nu(x) = 0 \tag{2.1.10}$$

where

$$(f(x), g(x))_X = f(x) \cdot g(x) \tag{2.1.11}$$

is the inner product of the Hilbert space elements $f(x)$ and $g(x)$ so that the square of the norm of the function f is

$$| f |_X^2 = \int_\Omega \{ f(x) \cdot f(x) \} \rho(x) d\nu(x) \tag{2.1.12}$$

If

$$F = \{ \phi_i : i \in \mathcal{I} \} \tag{2.1.13}$$

is a finite set of pairwise orthogonal functions in the space V of functions from Ω into the Hilbert space X, then

$$Pf(x) \;=\; \sum_{i\in\mathcal{I}} \left[\frac{\int_\Omega f(y)\cdot\phi_i(y)\rho(y)d\nu(y)}{\int_\Omega \phi_i(y)\cdot\phi_i(y)\rho(y)d\nu(y)}\right]\phi_i(x) \qquad (2.1.14)$$

The projection operator defined by equation (2.1.14) yields a generalized Fourier series approximation of functions; which is the basis of Mie like solutions of electromagnetic problems.

The next approximation scheme that is often used in electromagnetic analysis is the pulse basis function method. The pulse basis function method has been used by Guru and Chen [12], Hagmann and Gandhi [13], Hagmann and Levine [15], and Livesay and Chen [22] to predict the results of electromagnetic radiation with complex structures by decomposing the body into cells within each of which the induced electric vector is assumed to be a constant and charge densities are also assumed to be piecewise constant. The pulse basis function method makes use of the concept of the partition of an open set Ω of $\Re^n$.

We have defined for each x in $\Re^n$ and each positive number $r > 0$ the set

$$B(x,r) \;=\; \{y\in\Re^n \;:\; |x - y| \;<\; r\} \qquad (2.1.15)$$

to be the ball of radius r centered at x. We let Ω be an open set in $\Re^n$ whose closure is bounded.

Definition 2.1 *A partition of Ω is a set $\mathcal{P}(\Omega)$ of pairs (V_i, x_i) where $i\in\mathcal{I}$ and the ball, $B(x_i, r)$ is contained in V_i for some positive number r,*

$$\bigcup_{i\in\mathcal{I}} V_i \;=\; \Omega \qquad (2.1.16)$$

and

$$\mu_n(V_j \bigcap V_k) \;=\; 0 \;(j \neq k) \qquad (2.1.17)$$

whenever (V_i, x_i) and (V_j, x_j) are distinct members of the partition, $\mathcal{P}(\Omega)$ and μ_n is the standard Lebesgue measure on $\Re^n$ where we let

$$\mathcal{P}_1 \;=\; \{V_i : (V_i, x_i)\in\mathcal{P}(\Omega) \text{ for some } x_i\in V_i\} \qquad (2.1.18)$$

and we define the characteristic functions,

$$\chi_{V_i}(x) = \begin{cases} 1 & x \in V_i \\ 0 & x \text{ is not a member of } V_i \end{cases} \tag{2.1.19}$$

to be the characteristic functions or pulse functions associated with the sets V_i in $\mathcal{P}(\Omega)_1$. The sets V_i are called cells in a cellular decomposition of Ω.

Next we define the projection operators associated with this partition of an open set in Euclidean n dimensional space.

Definition 2.2 *We define the projection operator P associated with the partition,*

$$\mathcal{P}(\Omega) = \{(V_i, x_i) : x_i \in V_i\ ,\ i \in \mathcal{I},\ V_i \subset \Omega\} \tag{2.1.20}$$

by the rule,

$$Pf(x) = \sum_{V_i \in \mathcal{P}(\Omega)_1} [\chi_{V_i}(x) \cdot f(x_i)] \tag{2.1.21}$$

for all functions,

$$f : \Omega \to \mathbf{C}^m \tag{2.1.22}$$

where $\mathbf{C}^m$ denotes complex m dimensional space.

We prove the following.

Proposition 2.1 *If $\mathcal{F}(\Omega, \mathbf{C}^m)$ is any topological vector space of functions from Ω into $\mathbf{C}^m$ which includes all functions of the form,*

$$x \to \chi_V(x)\vec{u} \tag{2.1.23}$$

where

$$V \in \mathcal{P}(\Omega)_1 = \{W : (W, x) \in \mathcal{P}\} \tag{2.1.24}$$

and

$$\vec{u} \in \mathbf{C}^m \tag{2.1.25}$$

then the mapping P defined by equation (2.1.21) is an endomorphism of this topological vector space which satisfies

$$PP = P \tag{2.1.26}$$

2.2 The Standard but Nonoptimal Discretization

Kun Mu Chen (Livesay and Chen [22], Guru and Chen [12], and [25]) meticulously analyzed the electric field volume integral equation in the work he directed and assisted and correctly formulated the electric field volume integral equation for a nonmagnetic body as,

$$(\vec{E} - \vec{E}^i)(p,\omega) = \frac{i\omega^2}{c^2}\int_\Omega \left(\frac{(\epsilon-\epsilon_0)-i\sigma/\omega}{\varepsilon_0}\right)\overline{\overline{G}}(p,q)\cdot\vec{E}(q)dv(q) \tag{2.2.1}$$

where

$$\overline{\overline{G}} = \left(\overline{\overline{I}} + \left(\frac{1}{k_0^2}\right) grad(grad)\right)\left(\frac{exp(ik_0 \mid p-q \mid)}{4\pi \mid p-q \mid}\right) \tag{2.2.2}$$

What is done in practice is to apply the projection operator to the a priori unknown field $\vec{E}$ that appears under the integral and to also apply it also to both sides of the integral equation (2.2.1) to obtain the approximate equation

$$(P^x\vec{E} - P^x\vec{E}^i) = \frac{i\omega^2}{c^2}\int_\Omega \left(\frac{(\epsilon-\epsilon_0)-i\sigma/\omega}{\epsilon_0}\right)P^x\overline{\overline{G}}(x,y)\cdot(P^y\vec{E})dv(y) \tag{2.2.3}$$

where $\overline{\overline{G}}$ is defined by equation (2.2.2). The so called method of moments was developed in the early 1960s by mathematicians and is simply the weak topology approximation; as currently applied it is an attempt to do a better job of getting a more acceptable solution of the clearly nonoptimal approximation represented by equation (2.2.3). With the method of moments one obtains $3N$ equations for the $3N$ unknowns representing the electric vector in the N cells into which the scattering body Ω is decomposed by simply multiplying both sides of equation (2.2.3) by a function of x, often the characteristic function of the cell V_i, where i ranges from 1 to N, and integrating both sides of the new equation with respect to x

3 Exact Solutions of Integral Equations

We show in this section a method of creating a computerizable approximate to the original infinite rank integral equation. After multiplying all terms of the integral equation by the same invertible matrix, if necessary, we can reduce the coupled $\vec{E}$ and $\vec{H}$ integral equation to one of the form described in the following section.

3.1 Machine Precision in Integral Equation Methods

We show in this section how to correct our errors in an integral equation method, so that we can obtain, by doing more processing but not using excessive memory, an answer whose precision is close to that of the particular computing machine being used. Letting f be a vector valued function defined on an open set Ω of R^3 and having values belonging to a Banach space, X, which represents the set of values of the electric and magnetic field vector within the scattering body and having enough regularity that boundary values are defined. Suppose that the functions f that we consider all satisfy the condition,

$$f \in \mathcal{E}(\Omega, X), \tag{3.1.1}$$

that they belong to a Banach space of functions from Ω into X. We further suppose that we define a projection operator,

$$P : \mathcal{E}(\Omega, X) \rightarrow \mathcal{E}(\Omega, X) \tag{3.1.2}$$

We let $B(X)$ denote a Banach space of operators mapping X into itself and let K be a function,

$$K : \Omega \times \Omega \rightarrow B(X) \tag{3.1.3}$$

which in practice will represent the integral operator acting on the values of the electric and magnetic field vectors in the interior and on the surface of the scattering body. One way this can be handled is to assume enough regularity in the space of functions, $\mathcal{E}(\Omega, X)$ in which we are seeking the solution (and in the space of approximations within which we are attempting to find a solution that is reasonably close to actual solution), that the required boundary values are defined. Related to this basic projection operator, which

may be defined in one of the ways described in the previous section, or in other ways, we define the operator Q^x on functions from Ω into X by the rule,

$$P \int_\Omega K(x,y)(Pf)(y)d\nu(y) \;=\; \int_\Omega Q^x K(x,y)(Pf)(y)d\nu(y) \qquad (3.1.4)$$

We can reduce our original problem to that of solving an integral equation of the form,

$$f(x) - g(x) \;=\; \lambda T f(x) \qquad (3.1.5)$$

where

$$Tf(x) \;=\; \int_\Omega K(x,y)f(y)d\nu(y) \qquad (3.1.6)$$

and f may represent a two tuple consisting of the electric and magnetic vectors and g represents the result of applying an invertible linear transformation two a two tuple consisting of the electric and magnetic vectors of the incoming radiation. We define the operator L by the rule

$$L \;=\; PTPf(x) \qquad (3.1.7)$$

where P is a projection operator onto a space of approximates, and define the correction operator N by the rule,

$$Nf(x) \;=\; Tf(x) - Lf(x) \qquad (3.1.8)$$

Normally we require that P is a good enough approximator that solving the equation (2.2.3) will give us a satisfactory solution. However, with EFRIE theory we need only assume that P is good enough so that if N is defined by (3.1.8) that then the operator norm inequality,

$$max\,\{|\,\lambda\,||\,N\,|_\Omega\;,\;\;|\,\lambda\,|||\,(P-I)N\,|_\Omega\}\;<\;1 \qquad (3.1.9)$$

Thus, it follows that

$$T \;=\; L \;+\; N \qquad (3.1.10)$$

The usual approximate integral equation has the form

$$f_a \;=\; Pg \;=\; \lambda PTf_a \qquad (3.1.11)$$

where f_a satisfies the condition,

$$f_a \in P(\mathcal{E}(\Omega, X)), \qquad (3.1.12)$$

What is usually done is to assume that f_a is close enough to f to accurately represent the solution of the original infinite rank integral equation (3.1.5). We can, if inequality (3.1.9) is satisfied, define the bounded linear operator

$$G_\lambda = \sum_{k=1}^{n} \lambda^{k-1} N^k \tag{3.1.13}$$

so that it will follow that since formally and in fact,

$$(I - \lambda N) \cdot (I + \lambda N + \lambda^2 N^2 + \cdots) f = f \tag{3.1.14}$$

that by combining equations (3.1.13) and (3.1.14) that

$$(I - \lambda N)(I + \lambda G_\lambda) f = f \tag{3.1.15}$$

in view of the the geometric series relationship and the identity

$$(I + \lambda G_\lambda) = (I + \lambda N + \lambda^2 N^2 + \cdots) \tag{3.1.16}$$

for all functions f satisfying the relationship (3.1.1). Thus, we can in view of the relationship (3.1.10) deduce that

$$\lambda T = \lambda N + \lambda L \tag{3.1.17}$$

Equation (3.1.17) then means that we can express the original integral equation (3.1.5) in the form

$$f = g + \lambda N f + \lambda L f \tag{3.1.18}$$

Rearranging terms in equation (3.1.18) we see that

$$(I - \lambda N) f = g + \lambda L f \tag{3.1.19}$$

Combining equations (3.1.19) and (3.1.15) and equation (3.1.13) we deduce that

$$f = g + \lambda L f + \lambda G_\lambda (g + \lambda L f +) \tag{3.1.20}$$

Now if we simply combine equation (3.1.20) and equation (3.1.18) we deduce that

$$\lambda N f = \lambda G_\lambda (g + \lambda L f) \tag{3.1.21}$$

We would now like to apply the projection operator P to both sides of equation (3.1.20) making use of the fact that P is idempotent, equalling its square, and equation (3.1.7)

$$PL = L \tag{3.1.22}$$

to obtain the relation

$$Pf = Pg + \lambda Lf \lambda P(G_\lambda(g + \lambda Lf) \tag{3.1.23}$$

Substituting equation (3.1.21) into equation (3.1.23) we see that

$$Pf = Pg + \lambda Lf + \lambda PNf \tag{3.1.24}$$

Thus, if we define

$$L_{(K,P)} = PT \tag{3.1.25}$$

then in view of equation (3.1.7) and (3.1.25) we see that

$$Lf = L_{(K,P)}Pf \tag{3.1.26}$$

Now we see that equations (3.1.24) and (3.1.26) imply that

$$Pf = Pg + \lambda PL_{(K,P)}Pf + \lambda PNf \tag{3.1.27}$$

While equation (3.1.27) is not a finite rank integral equation, it suggests that an approximate finite rank integral equation

$$Pf_a = Pg + \lambda PL_{(K,P)}Pf_a + \lambda PNPf_a \tag{3.1.28}$$

might give a better approximation to the solution than the traditional approximation given by equation (3.1.11). We shall go much farther than this, however, and reduce the equation (3.1.24) to a true finite rank integral equation whose solution will be the projection Pf of the exact solution f of the original infinite rank integral equation (3.1.5) onto the space of approximates. This will permit us to achieve our ultimate objective of representing the solution f exactly in terms of Pf and the stimulating fields g by an exact formula. Going back to equation (3.1.19) and making use of equation (3.1.14) we obtain

$$f = \sum_{k=0}^{\infty} \left[(\lambda N)^k (g + \lambda Lf)\right] \tag{3.1.29}$$

Operating on both sides of equation (3.1.29) with N and then applying λP to both sides of this equation, we see that

$$\lambda PNf \;=\; \sum_{k=0}^{\infty}\left[(\lambda N)^k (g \;+\; \lambda Lf)\right] \tag{3.1.30}$$

Now, simply substituting equation (3.1.30) into equation (3.1.27) we get the finite rank integral equation,

$$\begin{aligned} Pf \;&=\; Pg + \\ \lambda PL_{(K,P)}Pf \;+\; \lambda PN &\left[\sum_{k=0}^{\infty}(\lambda N)^k \left(g \;+\; \lambda L_{(K,P)}Pf\right)\right] \end{aligned} \tag{3.1.31}$$

Now collecting terms in equation (3.1.31) involving Pf and those involving g and Pg we obtain the relationship

$$\begin{aligned} Pf \;=\; Pg \;+\lambda PN &\left[\sum_{k=0}^{\infty}(\lambda N)^k g\right] \;+ \\ \lambda P\left(L_{(K,P)} \;+\; N\left[\sum_{k=0}^{\infty}(\lambda N)^k\right]\lambda L_{(K,P)}\right) &Pf \end{aligned} \tag{3.1.32}$$

Our first objective is now achieved since equation (3.1.32) is a truly finite rank integral equation in the unknown member Pf of a finite dimensional vector space. The computer program giving a solution of equation (3.1.32) would provide us with coefficients of the basis vectors of this finite dimensional vector space that are needed to represent the solution Pf of equation (3.1.32). In other words, the linear combination of basis vectors of the vector space which is the image of the projector P is the exact value, Pf, of the projection of the exact value of the solution, f, of the original infinite rank integral equation, (3.1.5). From this point on we assume that Pf is known.

To finish off this section we use our exactly determined value of Pf that was obtained by solving equation (3.1.32) under the assumption that $I - \lambda L$ is invertible on the image of the projection operator P, where L is defined by equation (3.1.7). We begin by subtracting the right sides of equations (3.1.18) and (3.1.24) obtaining the relationship,

$$(f \;-\; Pf) \;=\; (g \;-\; Pg) \;+\; \lambda(I - P)Nf \tag{3.1.33}$$

Collecting the terms involving f in equation (3.1.33) an[illegible]ving the known function Pf over on the right side, we obtain the equation,

$$[I - \lambda(I - P)N]\,f \;=\; (g \;-\; Pg \;+\; Pf) \tag{3.1.34}$$

The inequality (3.1.9) then enables us to invert the operator acting on the f in the left side of equation (3.1.34) by applying the geometric series operator

$$S = \sum_{k=0}^{\infty}(\lambda(I-P)N)^k \tag{3.1.35}$$

to both sides of equation (3.1.34). Thus, once we solve equation (3.1.32) for Pf we can correct ourselves by expressing the exact value of f as

$$f = \sum_{k=0}^{\infty}(\lambda(I-P)N)^k \{g - Pg + Pf\} \tag{3.1.36}$$

Thus, without using auxiliary memory we can with a good enough start and enough iteration correct our solution to within computer accuracy.

4 Layered Materials

We have formulated some one dimensional scattering problems associated with magnetic materials, and solutions obtained from the differential equation formulations have been substituted into the integral equations and have been shown to satisfy them exactly. For magnetic materials, a single integral equation was obtained and the significance of surface values of the derivative of the electric vector were shown to be important. For higher order splines all terms arising in a matrix representation of the integral equation formulation of the problem, and all iterates of the integrals could be computed exactly. Using distribution theory concepts, we have combined the electric and magnetic field integral equations for the case of a plane wave that is incident normally on the magnetic slab.

4.1 MAGNETIC SLAB IE

We consider in this section radiation normally incident on a magnetic slab, and assume that the electric vector of the incident radiation has the form

$$E^i = E_0 \text{[illegible]} \tag{4.1.1}$$

so that the magnetic vector of the incident radiation defined by the Maxwell equation,

$$-i\omega\mu_0 H^i = curl(E^i) =$$

$$-e_y\left(-\frac{\partial}{\partial z}\right)E_0 exp(-ik_0 z)$$

$$= -ik_0 E_0 exp(-ik_0 z)e_y \tag{4.1.2}$$

is after dividing both sides of equation (4.1.2) by $-i\omega\mu$ is given by

$$H^i = \left(\frac{k_0 E_0}{\omega\mu_0}\right) exp(-ik_0 z)e_y \tag{4.1.3}$$

Within the magnetic slab, where the permittivity ϵ, the permeability μ, and the conductivity σ are diagonal tensors in Cartesian coordinates, the first Maxwell equation has the form,

$$curl(H) = (i\omega\epsilon_x + \sigma_x)E_x e_x + (i\omega\epsilon_y + \sigma_y)E_y e_y$$

$$+ (i\omega\epsilon_z + \sigma_z)E_z e_z \tag{4.1.4}$$

However, if the stimulating electric vector has only an x component, then the same is true of the reflected, induced, and transmitted radiation, and, thus, we may assume that within the slab that this is also true. Hence, we assume that within the slab,

$$E = g(z)exp(-i\omega t)e_x = E_x e_x \tag{4.1.5}$$

Since then

$$curl(E) = -e_y\left(-\frac{\partial}{\partial z}\right)E_x = -i\omega\mu_y H_y e_y \tag{4.1.6}$$

we conclude that

$$H_y = \frac{i}{\omega\mu_y}\frac{\partial E_x}{\partial z} \tag{4.1.7}$$

Using (3.4) we conclude that

$$curl(H) = e_x\left(-\frac{\partial}{\partial z}\right)H_y \tag{4.1.8}$$

which implies that

$$curl(H) =$$

$$e_x\left[\left(\frac{i}{\omega\mu_y^2}\mu_y^{(1)}(z)\right)\frac{\partial E_x}{\partial z} - \frac{i}{\omega\mu_y}\frac{\partial^2 E_x}{\partial z^2}\right]$$

$$= e_z(i\omega\epsilon_x + \sigma_x)E_x \qquad (4.1.9)$$

Thus, multiplying all terms of this last equation by $i\omega\mu_y$ we see that

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{\mu_y^{(1)}(z)}{\mu_y(z)}\frac{\partial E_x}{\partial z}$$

$$= (-\omega^2\mu_y\epsilon_x + i\omega\mu_y\sigma_x)E_x \qquad (4.1.10)$$

We are, therefore, seeking an impulse response of the equation,

$$\frac{\partial^2 E_x}{\partial z^2} + \omega^2\mu_0\epsilon_0 E_x =$$

$$\frac{\mu_y^{(1)}}{\mu_y}\frac{\partial E_x}{\partial z} + (\omega^2(\mu_0\epsilon_0 - \mu_y\epsilon_x) + i\omega\mu_y\sigma_x)E_x \qquad (4.1.11)$$

We introduce the variable

$$\tau = \omega^2\mu_y\epsilon_x - i\omega\mu_y\sigma_x - \omega^2\mu_0\epsilon_0, \qquad (4.1.12)$$

where we agree that ϵ, μ, and σ tak their free space values outside the slab, and assume that $E - E^i$ has the form,

$$E - E^i = c\int_{-\infty}^{\infty} \tau E_x exp(-ik_0 \mid z - \tilde{z} \mid)d\tilde{z}$$

$$+ b\int_{-\infty}^{\infty} \frac{\mu_y^{(1)}(\tilde{z})}{\mu_y(\tilde{z})}\frac{\partial E_x}{\partial \tilde{z}} exp(-ik_0 \mid z - \tilde{z}) \mid d\tilde{z} \qquad (4.1.13)$$

where we write the global magnetic permeability via the relationship

$$\mu_y(z) = (Y(z) - Y(z - L))(\tilde{\mu}_y - \mu_0) + \mu_0 \qquad (4.1.14)$$

where

$$Y(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases} \qquad (4.1.15)$$

is the Heaviside function and

$$Y^{(1)}(z) = \delta(z) \qquad (4.1.16)$$

is the Dirac delta function and where we think of μ as the permeability at any point and think of $\tilde{\mu}$ as the value of permeability inside the slab. Thus, with this definition and recognizing the tangential component of the magnetic field as being proportional to the

reciprocal of the the magnetic permeability times the derivative of the electric vector with respect to z in view of the relationship

$$H_y = \frac{i}{\omega\mu_y}\frac{\partial E_x}{\partial z}$$

and seek a representation of the form,

$$\begin{aligned}
E_x - E^i = c\int_0^L \tau E_x exp(-ik_0 \mid z-\bar{z}\mid d\bar{z} \\
+ b\int_0^L \frac{\mu_y^{(1)}(\bar{z})}{\mu_y(\bar{z})}\frac{\partial E_x}{\partial \bar{z}} exp(-ik_0 \mid z-\bar{z}\mid)d\bar{z} \\
+ b\left(1-\frac{\mu_0}{\mu_y(0)}\right)\frac{\partial E_x}{\partial z}(0)exp(-ik_0 z) \\
- \ b\left(1-\frac{\mu_0}{\mu_y(L)}\right)\frac{\partial E_x}{\partial z}(L)exp(ik_0 z)exp(-ik_0 L)
\end{aligned} \tag{4.1.17}$$

Theorem 4.1 *If E_x satisfies (4.1.17) and E_x is twice continuously at points inside and outside the slab, then (a) outside the slab $E - E^i$ has the representation*

$$E - E^i = \begin{cases} C^r exp(ik_0 z) & \textit{for } z < 0 \\ \\ C^t exp(-ik_0 z) & \textit{for } z > L \end{cases} \tag{4.1.18}$$

where C^r is the reflection coefficient, and C^t is the coefficient defining the transmitted radiation (c) if a function E_x that is differentiable inside and outside the slab satisfies the integral equation, then E_x is continuous on the entire real line, and furthermore, if $H - H^i$ is determined from (4.1.17) via the relationship

$$\begin{aligned}
H - H^i \ = \ \frac{-i}{2\omega\mu_0}\int_0^z \tau E_x exp(-ik_0(z-\bar{z})d\bar{z} \\
+ \frac{i}{2\omega\mu_0}\int_z^L \tau E_x exp(-ik_0(\bar{z}-z)d\bar{z} \\
+ \frac{i}{2\omega\mu_0}\int_0^z \frac{\mu_y^{(1)}(z)}{\mu_y}\frac{\partial E_x}{\partial \bar{z}} exp(-ik_0(z-\bar{z}))d\bar{z} \\
+ \frac{-i}{2\omega\mu_0}\int_z^L \frac{\mu_y^{(1)}(z)}{\mu_y}\frac{\partial E_x}{\partial \bar{z}} exp(-ik_0(\bar{z}-z))d\bar{z}
\end{aligned}$$

$$+ \frac{i}{2\omega\mu_0}\left(1 - \frac{\mu_0}{\mu_y(0)}\right)\frac{\partial E_z}{\partial z}(0)exp(-ik_0z)$$

$$+ \frac{i}{2\omega\mu_0}\left(1 - \frac{\mu_0}{\mu_y(L)}\right)\frac{\partial E_z}{\partial z}(L)exp(ik_0z)exp(-ik_0L) \tag{4.1.19}$$

and $H - H^i$ is continuous across the boundaries of the magnetic slab. Furthermore, the classical solutions of the integral equation (4.1.17) are solutions of Maxwell's equations provided that

$$b = \frac{i}{2k_0} \tag{4.1.20}$$

and

$$c = -\frac{i}{2k_0} \tag{4.1.21}$$

Proof. Equations (4.1.20) and (4.1.21), which represent the evaluation of the parameters in the integral equation (4.1.17) follows by substituting (4.1.17) into Maxwell's equations. We begin by computing the first and second partial derivatives of E_z with respect to z from the integral equations and we then use these expressions to show that (4.1.20) and (4.1.21) are needed in order that Maxwell's equations be satisfied. We find, upon breaking up the integral from 0 to L into the integral from 0 to z plus the integral from z to L and differentiating, that

$$\frac{\partial E}{\partial z} - \frac{\partial E^i}{\partial z} = c\tau E_z - c\tau E_z +$$

$$c(-ik_0) \int_0^z \tau E_z exp(-ik_0(z - \tilde{z}) d\tilde{z} +$$

$$c(ik_0) \int_z^L \tau E_z exp(-ik_0(\tilde{z} - z) d\tilde{z} +$$

$$b \frac{\mu_y^{(1)}}{\mu_y} \frac{\partial E_z}{\partial \tilde{z}} \mid_{\tilde{z}=z} - b \frac{\mu_y^{(1)}}{\mu_y} \frac{\partial E_z}{\partial \tilde{z}} \mid_{\tilde{z}=z} +$$

$$(-ik_0) b \int_0^z \frac{\mu_y^{(1)}(\tilde{z})}{\mu_y(\tilde{z}} \frac{\partial E_z}{\partial \tilde{z}} exp(-ik_0(z - \tilde{z}) d\tilde{z} +$$

$$(ik_0) b \int_z^L \frac{\mu_y^{(1)}(\tilde{z})}{\mu_y(\tilde{z}} \frac{\partial E_z}{\partial \tilde{z}} exp(-ik_0(\tilde{z} - z) d\tilde{z} +$$

$$(-ik_0) b \left(1 - \frac{\mu_0}{\mu_y(0)}\right) \frac{\partial E_z}{\partial z}(0) exp(-ik_0 z)$$

$$- \; (ik_0) b \left(1 - \frac{\mu_0}{\mu_y(L)}\right) \frac{\partial E_z}{\partial z}(L) exp(ik_0 z) exp(-ik_0 L) \qquad (4.1.22)$$

We now take the derivative of both sides of this last equation with respect to z obtaining

$$\frac{\partial^2 E}{\partial z^2} - \frac{\partial^2 E^i}{\partial z^2} =$$

$$c(-ik_0)^2 \int_0^z \tau E_z exp(-ik_0(z - \tilde{z})) d\tilde{z} + (-ik_0) c\tau E_z$$

$$- \; (ik_0) c\tau E_z + c(ik_0)^2 \int_z^L \tau E_z exp(-ik_0(\tilde{z} - z)) d\tilde{z} +$$

$$(-ik_0) b \frac{\mu_y^{(1)}(\tilde{z})}{\mu_y(\tilde{z})} \frac{\partial E_z}{\partial \tilde{z}} \mid_{\tilde{z}=z} + (-ik_0)^2 b \int_0^z \frac{\mu_y^{(1)}(\tilde{z})}{\mu_y(\tilde{z}} \frac{\partial E_z}{\partial \tilde{z}} exp(-ik_0(z - \tilde{z}) d\tilde{z} -$$

$$(ik_0) b \frac{\mu_y^{(1)}(\tilde{z})}{\mu_y(\tilde{z})} \frac{\partial E_z}{\partial \tilde{z}} \mid_{\tilde{z}=z} + (-ik_0)^2 b \int_z^L \frac{\mu_y^{(1)}(\tilde{z})}{\mu_y(\tilde{z}} \frac{\partial E_z}{\partial \tilde{z}} exp(-ik_0(\tilde{z} - z) d\tilde{z}$$

$$+ \; (-ik_0)^2 b \left(1 - \frac{\mu_0}{\mu_y(0)}\right) \frac{\partial E_z}{\partial z}(0) exp(-ik_0 z)$$

$$- \; (ik_0)^2 b \left(1 - \frac{\mu_0}{\mu_y(L)}\right) \frac{\partial E_z}{\partial z}(L) exp(ik_0 z) exp(-ik_0 L) \qquad (4.1.23)$$

We now make use of the fact that

$$-k_0^2(E - E^i) = -k_0^2 \left\{ c \int_0^L \tau E_z exp(-ik_0 \mid z - \tilde{z} \mid d\tilde{z} \right.$$

$$+ b\int_0^L \frac{\mu_y^{(1)}(z)}{\mu_y}\frac{\partial E_z}{\partial \bar{z}} exp(-ik_0 \mid z-\bar{z}\mid)d\bar{z}$$

$$+ b\left(1-\frac{\mu_0}{\mu_y(0)}\right)\frac{\partial E_z}{\partial z}(0)exp(-ik_0 z)$$

$$- b\left(1-\frac{\mu_0}{\mu_y(L)}\right)\frac{\partial E_z}{\partial z}(L)exp(ik_0 z)exp(-ik_0 L)\Big\} \qquad (4.1.24)$$

and substitute it into our equation for the difference between the second partial derivatives of the stimulated and incident electric field vectors. Rewriting (2.24) to make this substitution transparent we see that

$$\frac{\partial^2 E}{\partial z^2}-\frac{\partial^2 E^i}{\partial z^2}=$$

$$-(k_0)^2\Big\{c\int_0^z \tau E_z exp(-ik_0(z-\bar{z}))d\bar{z}$$

$$+ c\int_z^L \tau E_z exp(-ik_0(\bar{z}-z))d\bar{z}$$

$$+ b\int_0^z \frac{\mu_y^{(1)}(\bar{z})}{\mu_y(\bar{z}}\frac{\partial E_z}{\partial \bar{z}} exp(-ik_0(z-\bar{z})d\bar{z}$$

$$+ b\int_z^L \frac{\mu_y^{(1)}(\bar{z})}{\mu_y(\bar{z}}\frac{\partial E_z}{\partial \bar{z}} exp(-ik_0(\bar{z}-z)d\bar{z}$$

$$+ b\left(1-\frac{\mu_0}{\mu_y(0)}\right)\frac{\partial E_z}{\partial z}(0)exp(-ik_0 z)$$

$$- b\left(1-\frac{\mu_0}{\mu_y(L)}\right)\frac{\partial E_z}{\partial z}(L)exp(ik_0 z)exp(-ik_0 L)\Big\}$$

$$- 2(ik_0)c\tau E_z + 2(-ik_0)b\frac{\mu_y^{(1)}(z)}{\mu_y(z)}\frac{\partial E_z}{\partial z} \qquad (4.1.25)$$

Simplifying the above equation we find that

$$\frac{\partial^2 E}{\partial z^2}-\frac{\partial^2 E^i}{\partial z^2} = -k_0^2(E_z-E_z^i)$$

$$- 2cik_0\tau E_z - 2ik_0 b\frac{\mu_y^{(1)}(z)}{\mu_y(z)}\frac{\partial E_z}{\partial z} \qquad (4.1.26)$$

We next simplify this equation by making use of the fact that the electric vector, E_z^i, of the incident radiation satisfies the free space Helmholtz equation

$$\frac{\partial^2 E^i}{\partial z^2} + k_0^2 E^i = 0 \qquad (4.1.27)$$

Substituting this into the previous equation we find that

$$\frac{\partial^2 E_x}{\partial z^2} + k_0^2 E_x =$$

$$- 2cik_0 \tau E_x - 2ik_0 b \frac{\mu_y^{(1)}(z)}{\mu_y(z)} \frac{\partial E_x}{\partial z} \tag{4.1.28}$$

We now need to select c and b in the above equation so that the equation is identical to equation (4.1.11) where τ is given by

$$\tau = \omega^2 \mu_y \epsilon_x - i\omega \mu_y \sigma_x - \omega^2 \mu_0 \epsilon_0$$

$$= k^2 - k_0^2 = k^2 - \omega^2 \mu_0 \epsilon_0 \tag{4.1.29}$$

We see that we need only to require that

$$- 2ik_0 b = 1 \tag{4.1.30}$$

and

$$2ik_0 c = 1 \tag{4.1.31}$$

In order to define the operations we note here that, while it is true that we cannot in general multiply distributions, certain orders of distributions can act upon spaces larger than the infinitely differentiable functions. For example, order 0 distributions can act on the continuous functions with compact support, and order one distributions can act on the differentiable functions with compact support, et cetera which will enable us to define the product of an order 0 distribution u and a continuous function f by the rule,

$$(uf, \phi) = (u, f\phi) \tag{4.1.32}$$

where ϕ is a test function. However, the function uf is not a general distribution, but is a continous linear functional on the space of continous functions with compact support. The integral equation is then derived by recognizing that in view of equation (4.1.9) that

$$\frac{\partial^2 E_x}{\partial z^2} + k_0^2 E_x =$$

$$- i\omega \mu_y^{(1)}(z) H_y - \tau E_x \tag{4.1.33}$$

By convolving the fundamental solution of the left side of this equation with the right side we obtain the integral equation. Since, as we have shown ([7], [21]), every solution of the integral equation is a solution of Maxwell's equations and the solutions of the integral equation satisfy automatically the Silver Mueller radiation conditions and tangential components of the electric and magnetic vectors are automatically continuous across the boundaries, the solution of the integral equation is necessarily the solution of Maxwell's equations. Since the solution to this electromagnetic interaction problem is unique, the function space under consideration is the space of functions which are along with their derivatives continuous up to the boundaries. When the slab is nonmagnetic, then uniqueness may be proven in the function space ([21], pp 69-130) consisting of all vector valued functions ϕ such that

$$\int_{\Omega} | \phi |^2 \, dv + \int_{\Omega} | curl(\phi) |^2 \, dv < \infty \tag{4.1.34}$$

5 DISCRETIZATION

To approximate the integral equations on a computer with a finite memory, we divide the slab with which the radiation is interacting into thin wafers separated by planes whose normals are perpendicular to the planes defining the boundaries of the slab.

5.1 PIECEWISE LINEAR APPROXIMATION

We consider approximate integral equations of the form

$$\begin{aligned} \vec{E}(z) - \vec{E}^{i}(z) = \\ \sum_{j=1}^{N} \int_{z_{j-1}}^{z_j} \left\{ A_j + B_j(y - z_j^*) \right\} K(z,y) dy + \\ \sum_{j=1}^{N} \int_{z_{j-1}}^{z_j} B_j L(z,y) dy + \\ F(z) B_1 - G(z) B_N \end{aligned} \tag{5.1.1}$$

where we suppose that the numbers z_j are defined by

$$0 = z_0 < z_1 < \ldots < z_{j-1} < z_j < \ldots < z_N = L \tag{5.1.2}$$

and that within the subinterval (z_{j-1}, z_j), the electric vector is approximated by

$$\vec{E} = (A_j + B_j(z - z_j^*))\vec{e}_x, \tag{5.1.3}$$

where the constants A_j and B_j contain the $exp(i\omega t)$ time dependence. We have a separate equation for each value of z. At this stage there are several methods to obtain a matrix equation from this continuum of approximate equations. One obvious method is point matching by selecting two points ζ_{2j-1} and ζ_{2j} in the subinterval $[z_{j-1}, z_j]$. This gives us a system of $2N$ equations in $2N$ unknowns, which have the form

$$\begin{gathered}
E(\zeta_{2\ell-q+1}) - E^i(\zeta_{2\ell-q+1}) = \\
A_\ell + B_\ell(\zeta_{2\ell-q+1} - z_\ell^*) - E^i(\zeta_{2\ell-q+1}) = \\
\sum_{j=1}^{N} \int_{z_{j-1}}^{z_j} \left\{ A_j + B_j(y - z_j^*) \right\} K(\zeta_{2\ell-q+1}, y) dy + \\
\sum_{j=1}^{N} \int_{z_{j-1}}^{z_j} B_j L(\zeta_{2\ell-q+1}, y) dy + \\
F(\zeta_{2\ell-q+1}) B_1 - G(\zeta_{2\ell-q+1}) B_N
\end{gathered} \tag{5.1.4}$$

Defining

$$\delta_{(j,\ell)} = \begin{cases} 1 & j = \ell \\ 0 & j \neq \ell \end{cases} \tag{5.1.5}$$

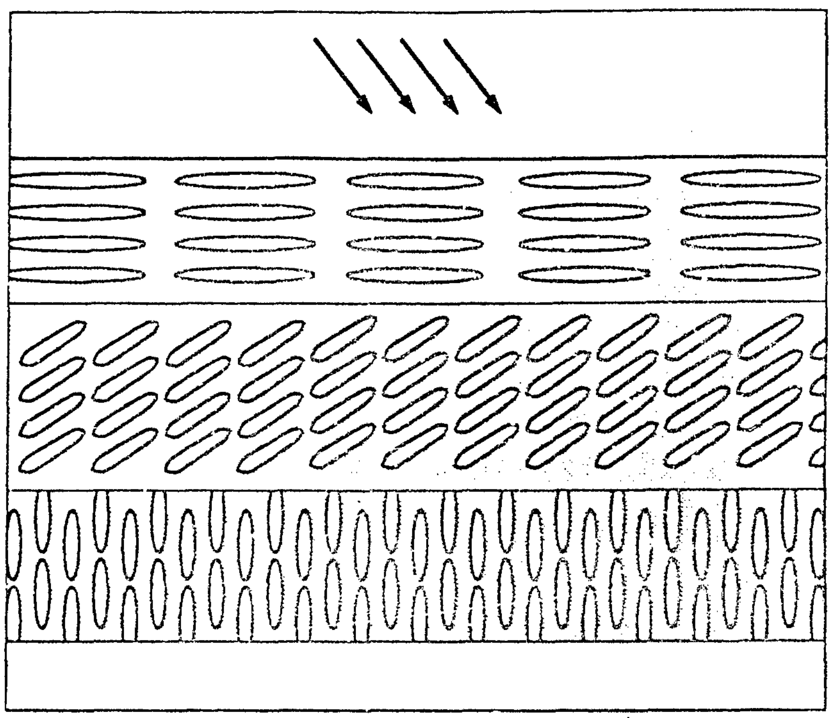
We now use the delta function notation to rewrite the previous equation to make it look like a matrix equation. We find that

$$\begin{gathered}
\sum_{j=1}^{N} \delta_{(j,\ell)} \left\{ A_j + B_j(\zeta_{2\ell-q+1} - z_\ell^*) \right\} - \\
-\sum_{j=1}^{N} \left\{ A_j \int_{z_{j-1}}^{z_j} K(\zeta_{2\ell-q+1}, y) dy \right. \\
\left. B_j \int_{z_{j-1}}^{z_j} (y - z_j^*) K(\zeta_{2\ell-q+1}, y) dy \right\} -
\end{gathered}$$

$$\sum_{j=1}^{N} \delta_{(j,1)} B_j F(\zeta_{2\ell-q+1}) +$$

$$\sum_{j=1}^{N} \delta_{(j,1)} B_j G(\zeta_{2\ell-q+1}) = E^i(\zeta_{2\ell-q+1}) \tag{5.1.6}$$

We now represent this last equation in the matrix form



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$$T\begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ A_N \\ B_N \end{pmatrix} = T\overline{\xi} = \begin{pmatrix} E^i(\zeta_1) \\ E^i(\zeta_2) \\ E^i(\zeta_3) \\ E^i(\zeta_4) \\ \cdot \\ \cdot \\ \vdots \\ E^i(\zeta_{2N-1}) \\ E^i(\zeta_{2N}) \end{pmatrix} \tag{5.1.7}$$

We now describe the entries of the matrix T. Note that if we define

$$\xi_{2j-1+p} = \begin{cases} A_j & p=0 \\ B_j & p=1 \end{cases} \tag{5.1.8}$$

that then the system of equations may be expressed more compactly in the form

$$\sum_{j=1}^{N}\left(\sum_{p=0}^{1} T_{(2\ell-1+q,2j-1+p)}\xi_{2j-1+p}\right) =$$

$$E^i(\zeta_{2\ell-q+1})) \tag{5.1.9}$$

where $q\epsilon\{0,1\}$. If $p=0$, then for each $q\epsilon\{0,1\}$ we have

$$T_{(2\ell-1+q,2j-1+p)} = \delta_{(j,\ell)} - \int_{x_{j-1}}^{x_j} K(\zeta_{2\ell-q+1},y)dy \tag{5.1.10}$$

On the other hand if $p=1$, then again for each $q\epsilon\{0,1\}$ we have

$$\begin{aligned} T_{(2\ell-1+q,2j-1+p)} &= \\ &\delta_{(j,\ell)}(\zeta_{2\ell-q+1} - z_\ell^*) + \\ &- \int_{x_{j-1}}^{x_j} K(\zeta_{2\ell-q+1},y)dy \\ &- \int_{x_{j-1}}^{x_j} L(\zeta_{2\ell-q+1},y)dy \\ &- \delta_{(j,1)}F(\zeta_{2\ell-q+1}) \\ &+ \delta_{(j,N)}G(\zeta_{2\ell-q+1}) \end{aligned} \tag{5.1.11}$$

Therefore, the solution of the matrix equation (5.1.7)

$$T\bar{\xi} = \vec{E}^i \tag{5.1.12}$$

then gives parameters in an approximate representation of the electric vector of the induced electromagnetic field.

6 Surface Integral Equation Methods

In this section we shall show how in the case where the irradiated structure consists of homogeneous regions which are delimited by diffeomorphisms of the interior of spheres in three dimensional space to represent the solution of the scattering problem as the solution of two combined field integral equations with integral operators formed from from the Green's functions defined on opposite sides of the separating surfaces. The surface integral equation methods reduce the computational complexity in the sense that they require discretization electric and magnetic fields defined on a surface rather than on a region of three dimensional space.

6.1 Combined Field Integral Equations

Consider a set Ω in $\mathbb{R}^3$ with boundary surface $\partial\Omega$ on which are induced electric and magnetic surface currents $\vec{J}_j$ and $\vec{M}_j$. If we have a simple $N+1$ region problem, where we have N inside and a region outside all N bounded homogenous aerosol particles corresponds to the region index j being equal to 1 and the region inside corresponds to j values ranging from 2 to $N+1$, then if the propagation constant k_j in region j is defined also by a function k_j, naturally defined on a Riemann surface as the square root of,

$$k_j^2 = \omega^2\mu\varepsilon - i\omega\mu\sigma \tag{6.1.1}$$

For a Debye medium (Daniel, [9]) the branch cuts are along the imaginary ω axis. For a Lorentz medium particle (Brillouin, [2], [31]) the branch cuts are in the upper half of the

complex ω plane parallel to the real axis. where μ, ϵ, and σ are functions of frequency that assure causality and that the radiation does not travel faster than the speed of light in vacuum. There are two Helmholtz equations, one for the interior of the particle and the other for the exterior, defined by

$$(\Delta + k_j^2)G_r = 4\pi\delta \tag{6.1.2}$$

where G_j is the temparate, rotationally invariant, fundamental solution ([18]) of the Helmholtz operator. We let

$$J_1 = J = -J_2 \tag{6.1.3}$$

and

$$M_1 = M = -M_2 \tag{6.1.4}$$

where we assume that the surface $S_{(1,2)}$ separates region 1 and region 2. We generalize equations (6.1.3) and (6.1.4) inductively by saying that for any surface $S_{(j,\bar{j})}$ separating region j from region $\bar{j}$ where

$$j < \bar{j} \tag{6.1.5}$$

we have

$$J_j = J = -J_{\bar{j}} \tag{6.1.6}$$

and

$$M_j = M = -M_{\bar{j}} \tag{6.1.7}$$

We define

$$\mathcal{I} = \left\{(j,\bar{j}) : S_{(j,\bar{j})} \text{ is a separating surface}\right\} \tag{6.1.8}$$

where j is less than $\bar{j}$. We get a single coupled, combined field integral equation which describes the interaction of radiation with the conglomerate aerosol particle or cluster given by

$$\vec{n} \times \vec{E}^{inc} = \vec{n} \times \sum_{(j,\bar{j})\in\mathcal{I}} \left\{ \left(\frac{i\omega}{4\pi}\right) \int_{S_{(j,\bar{j})}} \int \vec{J}(\bar{r}) \left(\mu_j \cdot G_j(r,\bar{r}) + \mu_{\bar{j}} \cdot G_{\bar{j}}(r,\bar{r})\right) da(\bar{r}) \right.$$

$$+ \frac{i}{4\pi\omega} grad \left\{ \int_{S_{(j,\bar{j})}} \int (div_s \cdot \vec{J}) \left[\frac{G_j(r,\bar{r})}{\epsilon_j} + \frac{G_{\bar{j}}(r,\bar{r})}{\epsilon_{\bar{j}}}\right] da(\bar{r}) \right\} +$$

$$\left(\frac{1}{4\pi}\right) curl \left(\int_{S_{(j,\bar{j})}} \int \vec{M}(\bar{r}) \cdot \left(G_j(r,\bar{r}) + G_{\bar{j}}(r,\bar{r}) \right) da(\bar{r}) \right) \Bigg\} \qquad (6.1.9)$$

In addition to equation (6.1.9) we need equation involving the magnetic vector $\vec{H}^{inc}$ of the stimulating electromagnetic field which is given by

$$\vec{n} \times \vec{H}^{inc} = \vec{n} \times \sum_{(j,\bar{j}) \in \mathcal{I}} \left\{ \left(\frac{i\omega}{4\pi}\right) \int_{S_{(j,\bar{j})}} \int \vec{M}(\bar{r}) \left(\epsilon_1 \cdot G_j(r,\bar{r}) + \epsilon_2 \cdot G_{\bar{j}}(r,\bar{r}) \right) da(\bar{r}) \right.$$

$$+ \left(\frac{i}{4\pi\omega}\right) grad \left\{ \int_{S_{(j,\bar{j})}} \int (div_s \cdot \vec{M}) \left[\frac{G_j(r,\bar{r})}{\mu_j} + \frac{G_{\bar{j}}(r,\bar{r})}{\mu_{\bar{j}}} \right] da(\bar{r}) \right\} +$$

$$\left. \frac{1}{4\pi} curl \left(\int_{S_{(j,\bar{j})}} \int \vec{J}(\bar{r}) \cdot \left(G_j(r,\bar{r}) + G_{\bar{j}}(r,\bar{r}) \right) da(\bar{r}) \right) \right\} \qquad (6.1.10)$$

Once the coupled combined field system (6.1.9) and (6.1.10) is solved for $\vec{J}$ and $\vec{M}$, the surface electric and magnetic currents respectively and we define the surface electric charge density by ([10], p 7)

$$\rho^e(\bar{r}) = \frac{i}{\omega} \left[div_s \cdot \vec{J}(\bar{r}) \right] \qquad (6.1.11)$$

and the surface magnetic charge density

$$\rho^m(\bar{r}) = \frac{i}{\omega} \left[div_s \cdot \vec{M}(\bar{r}) \right] \qquad (6.1.12)$$

where div_s is the surface divergence. Now for each region index j we define

$$\mathcal{J}(j) = \left\{ \bar{j} : (j,\bar{j}) \in \mathcal{I} \right\} \qquad (6.1.13)$$

where $\mathcal{I}$ is the set of all indices of separating surfaces defined by (6.1.8). We now need to be able to express the electric and magnetic fields inside and outside the scattering body. We first define the vector potentials $\vec{A}_j$ and $\vec{F}_j$ by the rules, ([10] [23])

$$\vec{A}_j = \sum_{\bar{j} \in \mathcal{J}(j)} \left[\frac{\mu_j}{4\pi} \int_{S_{(j,\bar{j})}} \int \vec{J}_j(\bar{r}) \cdot G_j(r,\bar{r}) da(\bar{r}) \right] \qquad (6.1.14)$$

$$\vec{F}_j = \sum_{\bar{j} \in \mathcal{J}(j)} \left[\left(\frac{\epsilon_j}{4\pi}\right) \int_{S_{(j,\bar{j})}} \int \vec{M}_j(\bar{r}) \cdot G_j(r,\bar{r}) da(\bar{r}) \right] \qquad (6.1.15)$$

The scalar potentials are defined in terms of the electric charge density (6.1.11) and magnetic charge density (6.1.12) by the rules,

$$\Phi_j(\bar{r}) = \sum_{\bar{j} \in \mathcal{J}(j)} \left[\left(\frac{1}{4\pi\epsilon_j}\right) \int_{S_{(j,\bar{j})}} \int \rho_j^e(\bar{r}) G_j(r,\bar{r}) da(\bar{r}) \right] \qquad (6.1.16)$$

and

$$\Psi_j(\bar{r}) = \sum_{\hat{j}\in\mathcal{J}(j)} \left[\left(\frac{1}{4\pi\mu_j}\right) \int_{S_{(j,\hat{j})}} \int \rho_j^m(\bar{r}) G_j(r,\bar{r}) da(\bar{r}) \right] \tag{6.1.17}$$

We now can define the electric and magnetic vectors inside the region j in terms of these potentials (6.1.14), (6.1.15), (6.1.16), and (6.1.17) by the rules,

$$\vec{E}_j = -i\omega\vec{A}_j(r) - grad(\Phi_j(r) + \frac{1}{\epsilon_j} curl(\vec{F}_j)(r) \tag{6.1.18}$$

and

$$\vec{H}_j = -i\omega\vec{F}_j(r) - grad(\Psi_j(r) + \frac{1}{\mu_j} curl(\vec{A}_j)(r) \tag{6.1.19}$$

Similar equations apply outside the body, by there the fields represented are the differences $\vec{E}_1^s$ and $\vec{H}_1^s$ between the total electric and magnetic vectors and the electric vector $\vec{E}^{inc}$ and the magnetic vector $\vec{H}^{inc}$ of the incoming wave that is providing the stimulation. Thus ([10]) we see that outside the body,

$$\vec{E}_1^s = -i\omega\vec{A}_1(r) - grad(\Phi_1(r) + \frac{1}{\epsilon_1} curl(\vec{F}_1)(r) \tag{6.1.20}$$

and

$$\vec{H}_1^s = -i\omega\vec{F}_1(r) - grad(\Psi_1(r) + \frac{1}{\mu_1} curl(\vec{A}_1)(r) \tag{6.1.21}$$

These equations generalize the formulation of Glisson ([10]) to a three dimensional structure whose regions of homogeneity of electromagnetic properties are diffeomorphisms of the interior of the sphere or a torus in R^3. If the scattering structure is not a body of revolution, then the region may be as general as a diffeomorph of an N handled sphere. These structures will be patched together to represent the organs of the body.

7 Contract Deliverables

The contractor shall deliver

- A program which uses EFRIE on a tissue slab to permit easy comparison of accuracy with the exact solution and with a weak topology convergence or method of moments analysis.

- A general volume integral equation formulation that will permit modeling of tissue heterogeneities, and an augmentation of the School of Aerospace Medicine Biomathematics Modeling Branch computer model of man with the moment method; this will just involve getting a different linear equation by multiplying both sides of the approximate equation by basis functions and integrating over the model of man numerically. The result will be a linear equation whose unknowns are parameters representing the electric field in each of the subunits into which the man was decomposed.

- A general surface integral equation formulation and computer implementation using the larger organs of the body, including the lungs, heart, liver, spleen, the major bones, and the stomach.

- A benchmarking of the surface integral equation approach with exact two layer sphere models with different tissue regions.

- A benchmarking of the surface integral equation approach with a program which describes the interaction of radiation with a structure delimited by N confocal spheroids

- The contractor shall deliver a best effort attempt to discover an algorithm for inverting N by N matrices in $N^{2+\epsilon}$ steps

Progress reports shall be delivered as progress is made along with all theory and ideas created by the contractor that are relevant to this effort that were created during the contract period. It is planned, because of the present market's greatly reduced PC cost it is suggested that we do the work on a 486 PC with a coprocessor chip and a high density

drive and to deliver the 486 PC with all created software, installed and ready to run, to the Brooks AFB sponsor at the end of this effort; at the sponsor's option this work may be done at the Brooks AFB site and may be coordinated with activities of or the scope of the effort may be augmented by the aid of regular Brooks AFB personnel.

8 Research Plan

A major portion of the effort involves the task of carrying out the details of the surface integral equation formulation of a detailed model of man by defining surfaces to represent the lungs, heart, liver, spleen, kidneys and other organs by assuming a tissue homogeneity within each separate organ or open space. We thereby reduce the computational complexity from an N^3 problem to an N^2 problem by requiring that we be able to estimate variations in magnetic and electric surface current within diffeomorphic images of triangular patches on the surface bounding the organ. We plan to use at least a polynomial of degree two in the surface variables, meaning that there will be 40 unknowns per cell. If we can use 1000 cells, we shall have 40 thousand equations. Rather than inverting a matrix, which would tax machine time, we shall simply use a quadratically converging iterative scheme (e. g. Newton's method) to solve the system of equations. The solution complexity is then only $\tilde{N}^2$, where $\tilde{N}$ is the number of equations. The coding shall be carried out on a PC and checked out for a two layer sphere, where the layers have different tissue properties, and the full program will be run on a government mainframe. The approximate solution Pf shall be obtained and corrected to machine precision using EFRIE techniques developed by the PI.

The key to spheroid scattering is to get the eigenvalues of the angular spheroidal harmonics and those of the more general spin weighted angular spheroidal harmonics. Two new novel approaches to this problem are proposed. One is a path following or homotopy method based on a Rayleigh Ritz functional. The second is a homotopy method based on a continued fraction expression involving ratios of the C_{2k} which eliminates from an infinite set of recursion relations defining the bounded solution of the angular spheroidal

harmonics ordinary differential equation everything except the a priori unknown eigenvalue which permits a series representation of the form

$$S = \eta^{\delta}(1-\eta^2)^{m/2}\sum_{k=0}^{\infty} C_{2k}\left(1-\eta^2\right)^k$$

that goes from an associated Legendre function representation to the angular spheroidal harmonic representation, since the the result of separating variables in spheroidal coordinates in the scalar Helmholtz equation yields the relationship

$$\left\{\frac{\partial}{\partial\eta}\left((1-\eta^2)\frac{\partial}{\partial\eta}S(c,\eta)\right)\right\}/S(c,\eta)$$
$$-\frac{m^2}{1-\eta^2} + c^2\eta^2 =$$
$$-\left\{\frac{\partial}{\partial\xi}\left((\xi^2+1)\frac{\partial R(c,\xi)}{\partial\xi}\right)\right\}/R(c,\xi) +$$
$$\frac{m^2}{\xi^2+1} + c^2\xi^2 = -\lambda_{(m,n)} \tag{8.0.22}$$

This yields an ordinary differential equation in $S(c,\eta)$ which is bounded at η equal to plus and minus one only for a discrete set of eigenvalues λ. If we substitute in the power series, we get a seemingly infinite set of recursion relations; a closer examination reveals that we can use continued fractions to eliminate the a priori unknown coefficients C_{2k} and get a single parameterized continued fraction expression for λ of the form

$$F(\lambda(s),n,m,c(s)) = 0 \tag{8.0.23}$$

where if

$$c(0) = 0 \tag{8.0.24}$$

the equation is that of the associated Legendre function $P_n^m(\eta)$ which means that

$$\lambda(0) = n\cdot(n+1) \tag{8.0.25}$$

Thus, the eigenvalues can be systematically determined, since $c(s)$ could be written as s times the actual value proportional to the distance between focal points of the spheroid, as the solution to the initial value problem

$$\lambda'(s) = \frac{D_4F(\lambda,n,m,c)c'(s)}{D_1F(\lambda,n,m,c)} \tag{8.0.26}$$

We simply solve the ordinary differential equation (8.0.26) to get to the eigenvalue, which then, because of the original recursion relationships gives us all the values of C_{2k}; the spheroidal harmonics are systematically determined, even when the material properties are complex. The spin weighted angular spheroidal harmonics can be determined by a similar method. This will provide a convincing benchmark for the general surface integral equation method.

Rapid matrix inversion efforts have been funded at Harvard and other mathematics institutions, but no one has been able to think of an algorithm. I propose to allow computer algebra packages to assist my thinking by looking at all possible sets of N^2 products, e.g. the set of all products of rows of the first matrix times columns of the second matrix with both ordinary and alternating sums; one element of the first matrix times the sum of all elements of the the second matrix which are not in the same row or column as this element. Continuing in this vein we create a set of products which have a cardinality which is a bounded multiple of N^2. If by a bounded set of sums of these products, we can create all entries of the product matrix, we shall have an algorithm for matrix multiplication which is of order $N^{2+\epsilon}$ if N is large enough. The appendix which proves this statement is attached to this proposal.

9 Potential Benefits

The fact that the results of this effort will provide the only machine precision integral equation formulation of electromagnetic interaction problems means that using carefully designed sources one could develop a means of focusing microwaves on a cancer tumor within the human body, and destroy the tumor by raising its temperature 4 degrees without harming the nearby normal tissue. This would bring the cost of harmless cancer treatment down to the level the working person.

The fact that this problem can be solved for anisotropic structures means that we have a practical means of optimally designing liquid crystal television sets and video displays, thereby making the home environment less hazardous for inner city children who spend so

much time with television and those who work with video display monitors. The reprogramming of these new safer devices will stimulate the economy by providing many new jobs.

The successful determination of an order $N^{2+\epsilon}$ matrix inversion algorithm, will assist us in the development of dynamical system models kinetic type models of world peace, which will include models that provide this nation and other nations with resource management plans for economic and ecological stability and medical care and food distribution as well as optimal running of coorporations and the design of very large systems. Also, the key part of the globally convergent homotopy method, with its myriad of design application potentials such the design of a magnet and irradiation configuration that would increase the absorption efficiency of a plasma in a fusion reactor by a factor of a billion or so, is the inversion of matrices.

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10 Budget

Any item in the budget below may be altered by the sponsor.

1206 hours	29.00 per hour	34,974.00
WCU overhead	20 percent off site	6994.00
configured 486 PC delivered to Brooks with above software installed and ready to run at contract end		3,000.00
totals		44968.00